GROUP ACTIONS AND ERGODIC THEORY

ISABEL LONGBOTTOM

1. Background and Definitions

A **topological group** is a group G endowed with a topology such that the multiplication map $\mu : G \times G \to G, (g,h) \mapsto gh$ and the inversion map $\iota : G \to G, g \mapsto g^{-1}$ are continuous.

A **representation** of a compact Hausdorff topological group G on a Banach (or Hilbert) space X is a map $\pi : G \to B(X)$ such that $\pi(1) = I$ and $\pi(gh) = \pi(g)\pi(h)$ for every $g, h \in G$. The representation is **weakly continuous** if $g \mapsto g \cdot x$ is continuous as a map $G \to (X, \tau_w)$ for every $x \in X$, or equivalently $g \mapsto \langle g \cdot x, x^* \rangle$ is continuous for every $x \in X, x^* \in X^*$. The representation is **strongly continuous** if $g \mapsto g \cdot x$ is continuous in the norm topology on X. A strongly continuous representation is automatically weakly continuous, since the weak topology is a coarser topology on the codomain. We write $\pi_g := \pi(g)$. A representation on a Hilbert space $\mathcal{H}$ is **unitary** if each π_g is a unitary operator on $\mathcal{H}$. (Over a real Hilbert space, one might also call such operators orthogonal. Each π_g preserves the inner product.)

A continuous map between Banach spaces with representations of a group G is called an **intertwining** operator if it commutes with the G-action. An isomorphism of representations is an intertwining operator with an inverse which is also intertwining. A closed subspace of X is called **reducing** if it is invariant under the G-action, and a reducing subspace gives a subrepresentation via restriction. A representation is **irreducible** if there are no proper nontrivial reducing subspaces.

On any compact Hausdorff topological group, there is a unique probability measure m that is invariant under left rotations (that is, under multiplication on the left by an element of the group). This measure is called the **Haar measure**, and it has the following properties:

- m is invariant under right rotations and inversion;
- m is strictly positive (the measure of any nonempty open set is nonzero).

The existence and uniqueness of the Haar measure is Theorem G.10 in [1].

Let G be a topological group and $\pi : G \to Y$ a multiplicative map, with Y a topological monoid. Then π is continuous if and only if it is continuous at $1 \in G$. This follows from the fact that G acts transitively on itself by multiplication. In particular, this holds when π is a representation and $Y = B(X)$.

In several places in this paper, we give proofs that use the sequence definition of continuity. This is not technically correct since a topological group G may not be metrizable. Nets can be used to adapt such arguments for general topological groups G.

2. Weak and Strong Continuity

We have the following result for Hilbert spaces, with a simple proof using the inner product.

Theorem 2.1. *(Proposition 1.1 of* [2]*) A unitary representation of a (compact, Hausdorff) group G is weakly continuous if and only if it is strongly continuous.*

Proof. Supose $\pi : G \to B(\mathcal{H})$ is weakly continuous. Then for every $v \in \mathcal{H}, g \in G$ we have

$$\|\pi_g(v) - v\|^2 = \|v\|^2 + \|\pi_g(v)\|^2 - 2\langle \pi_g v, v \rangle = 2\|v\|^2 - 2\langle \pi_g v, v \rangle$$

since π_g is unitary. Thus if $g \to 1$ in G then $\langle \pi_g v, v \rangle \to \|v\|^2$ by weak continuity so $\|\pi_g(v) - v\| \to 0$. This proves (strong) continuity at $1 \in G$, and thus π is a strongly continuous representation. $\square$

The rest of this section will be dedicated to upgrading this result to the case of a Banach space. Fix G a compact Hausdorff group with Haar measure m, and a weakly continuous representation $\pi : G \to B(X)$ on a Banach space X.

Definition 2.2. (Integral operators) For $f \in L^1(G)$ we can define an operator $\pi_f : X \to X^{**}$ by

$$\pi_f := \int_G f(g)\pi_g dg; \ \langle x^*, \pi_f(x) \rangle = \int_G f(g) \langle \pi_g(x), x^* \rangle dg$$

for $x^* \in X^*, x \in X$.

Then we have the estimate

$$\|\pi_f\|_{B(X,X^{**})} \leq \left(\sup_{g \in G} \|\pi_g\| \right) \cdot \|f\|_1.$$

This is finite because applying the uniform boundedness principle to the family of bounded functionals $(x^* \mapsto \langle x^*, \pi_g(x) \rangle)_{x \in X}$ gives $\sup_{g \in G} \|\pi_g\| < \infty$. This family is pointwise bounded on X^* because G is compact and $g \mapsto \langle \pi_g(x), x^* \rangle$ is continuous, so the set $\{\langle \pi_g(x), x^* \rangle; g \in G\}$ is compact in $\mathbb{R}$ for every $x^* \in X^*, x \in X$.

This construction a priori gives a map $X \to X^{**}$, but we will see that the element $\pi_f(x) \in X^{**}$ lies in $i(X)$, so we can treat π_f as a (linear) map from X to itself. Under this identification, the above estimate shows that $\pi_f \in B(X)$ for every $f \in L^1(G)$. In fact, we see that $\pi : L^1(G) \to B(X)$ is a bounded linear map. We state without proof two results which allow us to show $\pi_f(x) \in X$. The theorem of Banach is not used directly in the proof of Lemma 2.5, but is used to prove Kreĭn's theorem.

Theorem 2.3. *(Kreĭn, Thm C.11 in* [1]*) The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

Theorem 2.4. *(Banach, Thm G.8 in* [1]*) Let X be a Banach space and $x^{**} \in X^{**}$ such that its restriction to B_{X^*} is w^*-continuous. Then $x^{**} \in i(X)$ is represented by some $x \in X$.*

This theorem of Banach could be stated as an if and only if, since the weak* topology is the coarsest topology making every $x \in i(X)$ continuous on (X^*, τ_{w^*}). Also note that the theorem is trivial if X is reflexive.

We are now equipped to show that $\pi_f(x) \in X$.

Lemma 2.5. *(15.2 in* [1]*) For $f \in L^1(G)$, we have $\pi_f(x) \in X$ for every $x \in X$.*

Proof. Without loss of generality we may scale f so that $\int_G f = 1$. Moreover we may assume $f \geq 0$ by adding copies of $\mathbb{1}_G$, since the following proof shows $\pi_{\mathbb{1}}(x) \in X$. Define $K := \overline{\text{conv}}\{\pi_g(x); g \in G\} \subseteq X$. Then by Kreĭn's theorem, K is weakly compact, because $\{\pi_g(x); g \in G\}$ is the image of G under the map $g \mapsto \pi_g(x)$ and the image of a compact set under a weakly continuous map is weakly compact. Let $x^{**} \in X^{**} \setminus i(X)$. We will show $\pi_f(x) \neq x^{**}$, so $\pi_f(x) \in i(X)$. Since K is closed and convex, Hahn-Banach separation theorem gives that $\exists x^* \in X^*, \exists c \in \mathbb{R}$ with

$$\langle x^*, \pi_g(x)\rangle \leq c < \langle x^*, x^{**}\rangle \ \forall g \in G.$$

Since $f \geq 0$, multiplying the left inequality by $f(x)$ and integrating gives

$$\langle \pi_f(x), x^*\rangle = \int_G f(g)\langle x^*, \pi_g(x)\rangle dg \leq \int_G f(g)c \, dg = c$$

and so $\langle \pi_f(x), x^*\rangle < \langle x^*, x^{**}\rangle$. Thus $\pi_f(x) \neq x^{**}$. □

We will need one more lemma to show the equivalence of weak and strong continuity of representations. The next lemma shows that the set of elements of the form $\pi_f(x)$ is large in the sense that its span is dense in X. Together, Lemmas 2.5 and 2.6 will allow us to prove statements on X by considering only elements of the form $\pi_f(x)$ for continuous f.

Lemma 2.6. *For each $x \in X$, let $S_x := \{\pi_f(x); f \in C(G)\}$. Then $x \in \overline{S}^w$ and $\operatorname{span}\bigcup_{x \in X} S_x$ is dense in X.*

Proof. For the second part, let $x \in X$ with $(\pi_{f_n}(x))_{n\in\mathrm{N}}$ converging weakly to x. Then $\exists (y_n)_{n\in\mathrm{N}}$ in the convex hull of $(\pi_{f_n}(x))_{n\in\mathrm{N}}$ with $y_n \underset{n\to\infty}{\longrightarrow} x$ strongly. But the convex hull is contained in $\operatorname{span} S_x$, so $x \in \overline{\operatorname{span}}\, S_x$.

For the first part, first note that S_x is a subspace with $\pi_f(x)+\pi_h(x) = \pi_{f+h}(x)$ for $f, h \in C(G)$, and $\pi_{cf}(x) = c\cdot\pi_f(x)$ for $f \in C(G), c \in \mathbb{R}$. Thus S_x is convex and so by Mazur, $\overline{S_x} = \overline{S_x}^w$. We claim it is enough to check that every functional $x^* \in X^*$ vanishing on S_x also vanishes at x, since if $x \notin \overline{S_x}$ then by Hahn-Banach we can separate $\overline{S_x}$ and x to get $c \in \mathbb{R}$, $y^* \in X^*$ such that $\langle y^*, y\rangle \leq c < \langle y^*, x\rangle$ for every $y \in S_x$. Since S_x is a subspace, scaling $\langle y^*, y\rangle \leq c$ gives $\langle y^*, y\rangle = 0$ for every $y \in S_x$. By assumption this means $\langle y^*, x\rangle = 0$ which contradicts the strict inequality. Hence $x \in \overline{S_x}$.

To show that every functional vanishing on S_x vanishes at x, let $x^* \in X^*$ be such. Then

$$\int_G f(g)\langle \pi_g(x), x^*\rangle dg = 0 \ \forall f \in C(G)$$

so $\langle \pi_g(x), x^*\rangle = 0$ a.e. w.r.t. the measure m. But the Haar measure is strictly positive and $g \mapsto \langle \pi_g(x), x^*\rangle$ is continuous, so $\langle \pi_g(x), x^*\rangle$ is zero everywhere. Choosing $g = 1$ gives $\langle x, x^*\rangle = 0$ as we wanted. □

Before using the above to show that weak and strong continuity are equivalent for representations on a Banach space, we discuss an example which will be used in the proof.

Example 2.7. (Left and right regular representations) For $f \in L^1(G)$ and $g \in G$ we define $\tau_g \in B(L^1(G)), (\tau_g f)(h) = f(g^{-1}h) \ \forall h \in G$. Similarly $(R_g f)(h) = f(hg)$. Then $C(G) \subseteq L^1(G)$, so $\tau : G \to B(C(G)), g \mapsto \tau_g$ and $R : G \to B(C(G)), g \mapsto R_g$ are the left and right **regular representations** respectively. In particular, these representations are both (strongly) continuous.

Since the Haar measure is left and right rotation invariant, the left and right regular representations are well-defined on $L^p(G)$ for each $1 \leq p \leq \infty$. Continuity of the regular representations on $L^p(G)$ follows from continuity on $C(G)$ and density of $C(G)$ in $L^p(G)$.

We are now in a position to show equivalence of weak and strong continuity, using Lemma 2.6 and the fact that the left regular representation on $L^1(G)$ is continuous.

Theorem 2.8. *Let $\pi : G \to B(X)$ be a representation of a compact Hausdorff group G on a Banach space. The following are equivalent:*

(i) π is weakly continuous;
(ii) π is strongly continuous;

(iii) the map $G \times X \to X, (g, x) \mapsto g \cdot x$ is continuous.

If any (and therefore all) of the above hold then the weak and strong operator topologies on $\pi_G = \{\pi_g; g \in G\}$ are the same.

Proof. For (i) $\implies$ (ii), define $F = \{x \in X; g \mapsto \pi_g(x) \text{ is continuous}\}$. This is a subspace, and it is closed. (An easy argument using sequences and the uniform boundedness of the operators π_g shows that it is closed when G is metrizable.) Now for $f \in L^1(G)$ and $g \in G$, a computation shows

$$\langle \pi_g \pi_f(x), x^* \rangle = \int_G f(g^{-1}h) \langle \pi_h x, x^* \rangle dh$$

for every $x \in X, x^* \in X^*$. Hence $\pi_g \pi_f = \pi_{\tau_g(f)} \in B(X)$. Now $f \mapsto \pi_f(x)$ and $g \mapsto \tau_g(f)$ are continuous, by continuity of $\pi : L^1(G) \to B(X)$ and τ respectively. Hence the composition $g \mapsto \pi_{\tau_g(f)}(x) = \pi_g \pi_f(x)$ is continuous $\forall f \in L^1(G), x \in X$. Thus $\mathrm{R}(\pi_f) \subseteq F$. By Lemma 2.6 we know $\operatorname{span} \mathrm{R}(\pi_f)$ is dense, so $F = X$.

Then (ii) $\implies$ (iii) follows from uniform boundedness of the operators π_g and a sequence (or net) convergence argument, and (iii) $\implies$ (i) is immediate.

For the last part, assume (ii) holds. Then π_G is compact in the strong operator topology, and therefore also compact in the weak operator topology. Then the identity map $(\pi_G, \tau_s) \to (\pi_G, \tau_w)$ is continuous between compact Hausdorff spaces, so must be a homeomorphism. Hence the weak and strong topologies coincide on π_G. □

3. Decomposing Representations

Definition 3.1. The **convolution** $f * k$ for $f, k \in L^1(G)$ is

$$f * k := \tau_f(k) = \int_G f(g) \tau_g k \; dg.$$

If $k \in C(G)$ then one can evaluate pointwise to get $(f * k)(h) = \int_G f(g) k(g^{-1}h) dg$.

We next state some properties of unitary representations, which can be used to more generally characterise representations on Banach spaces. The slogan for this discussion is that an irreducible unitary representation of a compact group on a Hilbert space must be finite-dimensional, and so a representation on a Banach space can be fully dscribed by its restriction to certain finite-dimensional subspaces where the representation is irreducible and the group elements act unitarily.

Lemma 3.2. *(Properties of Unitary Representations) Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a compact group on a Hilbert space. Then the following properties hold:*

(i) A closed subspace F is reducing if and only if its orthogonal complement $F^\perp$ is reducing.
(ii) A closed subspace F is reducing if and only if the orthogonal projection onto F is an intertwining operator.
(iii) If $\mathcal{H}$ is finite-dimensional then π decomposes orthogonally into irreducible representations.

Part (iii) follows from (i) by induction, since we can write a reducible representation as the direct sum of its restriction to some reducing subspace and the orthogonal complement. Part (i) follows from the representation being unitary, so $\pi_G = \{\pi_g; g \in G\}$ is closed under taking adjoints.

Definition 3.3. (Coordinate functions) Up to a choice of orthonormal basis, a unitary representation of G on a finite-dimensional Hilbert space is a group homomorphism $\psi : G \to O(n)$ into the matrix group of orthogonal $n \times n$ matrices. A continuous function $f \in C(G)$ is a **coordinate function** if there exists a continuous unitary representation $\psi : G \to O(n)$ with $f = \psi_{ij}$. Equivalently, if f is of the form $f(g) = \langle \pi_g e_i, e_j \rangle$ for a continuous representation π and orthonormal basis (e_j). A coordinate function is called irreducible if it comes from an irreducible representation.

A central result for coordinate functions on a Hilbert space is the following. It is proved using the Stone-Weierstrass theorem, and can be used to bootstrap more general results for coordinate functions on Banach spaces, as we will see.

Theorem 3.4. *(Thm 15.8 in* [1]*) For a compact Hausdorff group G, the linear span of the irreducible coordinate functions is dense in $C(G)$.*

Theorem 3.5. *(Stone-Weierstrass, Thm 4.4 in* [1]*) Let K be a nonempty compact Hausdorff topological space.*

(a) ($\mathbb{R}$-version) Let A be a subalgebra of $C(K, \mathbb{R})$ which contains a constant function and separates points. Then A is dense in $C(K, \mathbb{R})$.
(b) ($\mathbb{C}$-version) Let A be a subalgebra of $C(K, \mathbb{C})$ which is invariant under complex conjugation and separates points. Then A is dense in $C(K, \mathbb{C})$.

The proof of Theorem 3.4 is by application of Stone-Weierstrass. Note the following simplifications. First, any finite-dimensional representation decomposes into irreducible representations, so by choosing an orthonormal basis consistent with such a decomposition, we see that every coordinate function is a linear combination of irreducible coordinate functions. So it is enough to show that the span of the coordinate functions is dense. Second, the product of two coordinate functions is a coordinate function of the tensor product of their corresponding representations, so the coordinate functions are closed under multiplication. Similarly, the complex conjugate of a coordinate function over $\mathbb{C}$ is a coordinate function of the dual representation. Thus the span of the coordinate functions forms a subalgebra of

$C(G)$. Finally, the constant function $\mathbb{1}$ is a coordinate function of the trivial representation ($\pi_g = I\ \forall g \in G$) so it is left to show only that the coordinate functions separate points.

The following proposition will allow us to do so.

Proposition 3.6. *(15.12 in* [1]*) Let $f \in L^2(G)$ be nonzero. Then there exists a finite-dimensional unitary representation $\pi : G \to B(\mathcal{H})$ such that $\pi_f \neq 0$.*

The proof of this proposition follows by defining $k = f * f^*$ and considering the operator on $L^2(G)$ given by $u \mapsto u * k$. This operator is compact, self-adjoint and intertwining with respect to the left regular representation by properties of the convolution, so the spectral theorem allows us to decompose $\mathcal{H}$ into eigenspaces. These eigenspaces are invariant under left-rotation and the action of f is nonzero on some eigenspace. Then the desired representation is the left regular representation restricted to such an eigenspace.

Using this proposition, we can do the following. Suppose $g, h \in G$ with $g \neq h$. To separate g and h, we want to find a coordinate function f such that $f(g) \neq f(h)$. This is equivalent to finding a finite-dimensional unitary representation π such that $\pi_g \neq \pi_h$. Pass to gh^{-1}, so we may assume $g \neq 1$ and we want to find $\pi_g \neq I$, that is a representation π where g does not act trivially. Let V, V' be disjoint open neighbourhoods of $1, a \in G$ respectively, and consider $U := V \cap a^{-1}V'$. This is an open set containing 1, and $U \subset V$ while $aU \subset V'$ so $U \cap aU = \emptyset$. Then there exists a nonnegative continuous function $0 \leq u \in C(G)$ with $u(1) > 0$ and $\operatorname{supp}(u) \subseteq U$. Hence $f := u - \tau_a(u)$ is nonzero (in particular, at 1), so by Proposition 3.6 there exists a finite-dimensional unitary representation π with $\pi_f \neq 0$. Then

$$0 \neq \pi_f = \pi_{u - \tau_a u} = \pi_u - \pi_{\tau_a u} = \pi_u - \pi_a \pi_u$$

so $\pi_u \neq \pi_a \pi_u$ and thus $\pi_a \neq I$. This proves Theorem 3.4.

We next wish to apply the Hilbet space results to a Banach space representation in order to characterise representations on Banach spaces in terms of their restrictions to finite-dimensional reducing subspaces. Let $\pi : G \to B(X)$ be a strongly (or equivalently, weakly) continuous representation on a Banach space. A finite linearly independent set $(e_j)_{j=1}^n$ in X is called a **unitary system** for π if the subspace $F := \operatorname{span}\{e_1, \dots, e_n\}$ is G-invariant and the matrix representation $\chi : G \to \mathbb{R}^{n \times n}$ (or $\chi : G \to \mathbb{C}^{n \times n}$) defined by $\pi_g e_i = \sum_{j=1}^n \chi_{ij}(g) e_j$ is unitary. That is, $\pi|_F$ is unitary with respect to the inner product on F given by declaring $(e_j)_{j=1}^n$ to be an orthonormal basis. A unitary system is irreducible if $\pi|_F$ is.

Remark 3.7. It is a result from Analysis 1 (or see [3]) that any two norms on a finite dimensional vector space (over $\mathbb{R}, \mathbb{C}$, or any valued field) are equivalent and so define the same topology. Therefore the innner product we have constructed on F by nominating an orthonormal basis induces the subspace topology of $F \subseteq X$. However, the norm on X restricted to F may not be the same as that induced by the inner product, it is only the same up to a bound by a constant. This will not worry us, since we define a unitary system in terms of the chosen inner product.

We have the following useful results for unitary systems.

Lemma 3.8. *(15.13 in* [1]*) For a continuous representation π, any finite-dimensional π_G-invariant subspace has a corresponding unitary system, and the finite-dimensional space* $\operatorname{span}\{\pi_{\chi_{ij}} u; 1 \leq i, j \leq n\}$ *is π_G invariant for any unitary representation $\chi : G \to O(n)$.*

The proof of the first part of the lemma follows from the fact that $\pi_g \pi_{\chi_{ij}} x = \pi_{\tau_g \chi_{ij}} x = \sum_{k=1}^n \chi_{ik}(g^{-1}) \pi_{\chi_{kj}} x$. The second part follows by taking any inner product $\langle \cdot, \cdot \rangle_1$ on F, and noting that the average over G, $\langle u, v \rangle := \int_G \langle \pi_g u, \pi_g v \rangle_1 dg$ is still an inner product. Then π_g acts isometrically with respect to this inner product because the Haar measure is rotation invariant, so any corresponding orthonormal basis for F is a unitary system.

Theorem 3.9. *Let π be a continuous representation of a compact Hausdorff group G on a Banach space X. Then*

$$X = \overline{\bigcup}^{w} \{F; F \textit{ is a } \pi_G\textit{-invariant closed susbapce}, \dim F < \infty\} = \overline{\operatorname{span}}\ \{e_j; (e_j)_{j=1}^n \textit{ is a unitary system}, 1 \leq j \leq n\}.$$

That is, the union of all finite-dimensional reducing subspaces is weakly dense, and the span of all unitary systems is dense.

Proof. The second equality follows from Lemma 3.8 and the fact that unitary systems can be decomposed (orthogonally) into irreducible subspaces. For the first equality, by Theorem 3.4 the span of coordinate functions of finite-dimensional unitary representations of G is dense in $C(G)$. Using boundedness of the map $f \mapsto \pi_f, L^1(G) \to B(X)$ we can take a sequence in the linear span of the coordinate functions limiting to $f \in C(G)$, and then the image of this sequence under the map $k \mapsto \pi_k$ evaluated at $x \in X$ converges to $\pi_f x$. Hence $\pi_f x$ lies in the closure of the linear span of irreducible unitary systems. But we already know elements of the form $\pi_f x$ form a w-dense subset of X by Lemma 2.6 so

$$\overline{\bigcup}^{w} \{F; F \text{ is a } \pi_G\text{-invariant closed susbapce}, \dim F < \infty\} \supseteq \overline{\{\pi_f x; f \in C(G), x \in X\}}^{w} = X.$$

$\square$

Remark 3.10. It follows that every infinite-dimensional continuous representation on a Banach space has a nontrivial finite-dimensional invariant subspace, otherwise the union on the right hand side of Theorem 3.9 would be empty. In particular, an irreducible unitary representation on a Hilbert space must be finite-dimensional, since it cannot have any finite-dimensional invariant subspaces. The hypothesis that the representation be unitary here is needed so that the inner product on any unitary system can be taken to be the restriction of the inner product on the original space, so that a basis for any unitary system is orthogonal in the inner product on $\mathcal{H}$.

4. Markov and Ergodic Representations

We next consider a particular class of measure-preserving representations on $L^1(X)$ for a probability space X. Such representations are particularly nice because they can be characterised by a group acion on the underlying topological space, and properties of the representation can be measured from this group action.

Definition 4.1. A **Markov representation** is a representation on a Banach space of the form $L^1(X)$ for some propability space X, where each π_g is a measure-preserving operator (called a Markov embedding) in the following sense:

- $\pi_g \mathbb{1} = \mathbb{1}$;
- $\pi_g \geq 0$, meaning that if $f \in^1 (X), f \geq 0$ a.e. on X then $\pi_g f \geq 0$ a.e. on X also;
- $\int_X \pi_g f = \int_X f$ for all $f \in L^1(X)$;
- $|\pi_g f| = \pi_g |f|$ for all $f \in L^1(X$.

Note that the last two properties together imply that $\|\pi_g f\|_1 = \|f\|_1$ so each π_g is an isometry.

We now see a motivating class of Markov representations.

Example 4.2. Suppose we have a continuous group action $X \times G \to X, (x, g) \mapsto x \cdot g$ where X is a (compact Hausdorff) probability space and G is some topological group. If this action is measure-preserving, then it gives rise to a Markov representation on $L^1(X)$ via $(\pi_g f)(x) := f(x \cdot g)$. The first property above follows from the fact that the action of each $g \in G$ on X is bijective. The second and third properties follow because the G-action is measure-preserving.

Two G-actions are isomorphic if there is a homeomorphism between the spaces they act on which respects the G-action. In this case they give rise to isomorphic representations.

A Markov representation is called **ergodic** if the common fixed points of the G-action are precisely the constant functions, that is

$$\operatorname{fix} \pi_G := \bigcap_{g \in G} \operatorname{fix} \pi_g = \mathbb{R}\mathbb{1}_X \text{ or } \mathbb{C}\mathbb{1}_X. \tag{4.1}$$

Example 4.3. (Homogeneous systems) Let G be a compact Hausdorff group and H a closed subgroup. The Haar measure on G induces a probability measure on $H \setminus G$, the space of right cosets, which is invariant under the canonical action of G by multiplication. The G-action is continuous, with the topology on $H \setminus G$ the quotient topology with respect to the surjection $G \to H \setminus G$. This is called a homogeneous action. Thus as in the previous example, we obtain a Markov representation of G on $L^1(H \setminus G)$. A representation that arises in this way is called a **homogeneous system**. A homogeneous system is in particular ergodic.

We now take a slight detour and consider the induced representation of G on $C(K)$ for a continuous (right) group action on a compact set K. For such an action, since G is compact each orbit $x \cdot G, x \in K$ is compact and G-invariant. Any two orbits are disjoint or equal. Hence the associated representation decomposes into the direct sum of irreducible representations on the orbits, of the form $\pi|_{x \cdot G} : G \to B(C(x \cdot G))$.

For any point $x \in K$, the preimage of the closed set $\{x\}$ under the (continuous) map $g \mapsto g \cdot x$ is a closed subgroup of G, called the **stabiliser** G_x. We know G acts on both $G_x \setminus G$ and $x \cdot G$ via multiplication, and there is a homeomorphism between these spaces given by $\varphi(G_x \cdot g) = x \cdot g$. This is an isomorphism of G-actions, so the induced representation on the orbit $x \cdot G$ is isomorphic to that arising from the homogeneous action corresponding to G_x.

In particular, the induced representation on $C(K)$ corresponding to a G-action on a compact space is irreducible if and only if the action is isomorphic to a homogeneous action. This is is also equivalent to the fixed space of the G-action being trivial (cf the definition of ergodicity, 4.1). Conversely, we have the following theorem, which tells us that every algebraic representation of G on $C(K)$ arises from a topological action on K.

Theorem 4.4. *Let G be a group, K a compact Hausdorff space and $\pi : G \to B(C(K))$ a representation where each operator π_g is an algebra homomorphism (respects addition and multiplication of continuous functions). Then π arises as the representation associated with a topological group action of G on K. This action is continuous if and only if the representation π was continuous.*

We now return to our discussion of Markov representations. We have the following two key results relating homogeneous systems, Markov representations and compact group actions.

Theorem 4.5. *(15.27 in* [1]*) Let $\pi : G \to B(L^1(X))$ be a continuous Markov representation of a compact Hausdorff group G. Then there exists a compact probability space K with continuous measure-preserving G-action such that the induced representation on $L^1(K)$ is isomorphic to π. Moreover, if $L^1(X)$ is separable then K can be chosen to be metrizable.*

Theorem 4.6. *(15.28 in* [1]*) For a continuous Markov representation $\pi : G \to B(L^1(X))$ of a compact Hausdorff group G, the representation π is ergodic if and only if it is isomorphic to a homogeneous system.*

We therefore see that every Markov representation arises from a topological group action, and such a representation is ergodic precisely when the associated action is homogeneous.

5. References

[1] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator theoretic aspects of ergodic theory*, vol. 272. Springer, 2015.
[2] B. Casselman, *Unitary representations*. University of British Columbia. https://www.math.ubc.ca/~cass/research/pdf/Unitary.pdf.
[3] K. Conrad, *Equivalence of Norms*. University of Connecticut. https://kconrad.math.uconn.edu/blurbs/gradnumthy/equivnorms.pdf.