K-THEORY OF THE K(1)-LOCAL SPHERE VIA TC

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Abstract. We state and give proofs of the key theorems from [Lev22], as well as explaining the Land-Tamme $\odot$ construction, introduced in [LT19] and used in a crucial way to obtain these results. This is in aid of understanding $K(L_{K(1)}\mathbb{S})$, the height 2 counterexample to the telescope conjecture. These notes are from a lecture given in the Harvard Thursday Seminar in February 2024, about the recent paper [BHLS23] disproving the telescope conjecture for all primes at heights $n \geq 2$.

Contents

1. Introduction and main results

This lecture follows [Lev22] and [LT19]. Preliminaries include higher algebra (particularly Chapter 7.1 in [Lur17]) and algebraic K-theory. Context in the Thursday Seminar: The counterexample to the telescope conjecture at height 2 is $K(L_{k(1)}\mathbb{S})$.

In this talk, we will see how to compute $K(L_{k(1)}\mathbb{S})$, at least in terms of TC and some other things we can understand. Recall that after $T(n)$-localising for $n \geq 2$, we have $L_{T(n)}K(L_{k(1)}\mathbb{S}) \cong L_{T(n)}K(\ell^{h\mathbb{Z}})$ where ℓ is the (p-complete) Adams summand[1] of connective topological K-theory, with $\mathbb{Z}$-action via the Adams operation ψ^{p+1}. Concretely, we get:

Theorem A (Levy, $p > 2$)**.** $K(L_1^f\mathbb{S}_p) \cong K(L_{K(1)}\mathbb{S})$, there is a cofibre sequence split on π_*

$$K(\ell^{h\mathbb{Z}}) \longrightarrow K(L_{K(1)}\mathbb{S}) \longrightarrow \Sigma K(\mathbb{F}_p)$$

and a pullback square

$$\begin{array}{ccc} K(\ell^{h\mathbb{Z}}) & \longrightarrow & \mathrm{TC}(\ell^{h\mathbb{Z}}) \\ \downarrow & \lrcorner & \downarrow \\ K(\mathbb{Z}_p) & \longrightarrow & \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}). \end{array}$$

Let F be the fibre of the map $\mathrm{TC}(\ell^{h\mathbb{Z}}) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}})$. Then $F[\frac{1}{p}] = 0$, F is $(2p-2)$-connective and $\pi_{2p-2}(F/p) \cong \bigoplus_0^\infty \mathbb{F}_p$.

Remark 1.1. What does this theorem get us?

[1] We are assuming for simplicity that $p \neq 2$. Everything can be made to work at the prime 2 with ℓ replaced by ko; for details of the necessary changes see [Lev22].

(1) Looking $T(n)$-locally for $n \geqslant 2$,

$$L_{T(n)}K(L_{K(1)}\mathbb{S}) \simeq L_{T(n)}K(\ell^{h\mathbb{Z}}) \overset{Cor. 4.1}{\simeq} L_{T(n)}\,\mathrm{TC}(\ell^{h\mathbb{Z}}).$$

So we have reduced the computation we need for the disproof of the telescope conjecture at height 2 to something about TC.

(2) After inverting p, the vertical fibres vanish. So $K(\ell^{h\mathbb{Z}})[\frac{1}{p}] \simeq K(\mathbb{Z}_p)[\frac{1}{p}]$ can be computed. The split cofibre sequence on π_* then allows us to compute $\pi_* K(L_{K(1)}\mathbb{S})[\frac{1}{p}]$ in full.

(3) To understand $K(L_{K(1)}\mathbb{S})$ we just need to understand $K(\mathbb{F}_p)$, $K(\mathbb{Z}_p)$, and the fibre $F = \mathrm{fib}(\mathrm{TC}(\ell^{h\mathbb{Z}}) \to \mathrm{TC}(\mathbb{Z}_p^{h\mathbb{Z}}))$.

The only part of this theorem that we will use in the rest of the seminar is the pullback square. This is an ingredient in the disproof of the telescope conjecture at height 2, see [BHLS23]. To prove Theorem A, the main technical result we are going to need is a version of [DGM12] for (-1)-connective rings.

Theorem B (Levy)**.** Let $f : R \to S$ be a map of connective $\mathbb{E}_1$-rings with a $\mathbb{Z}$-action such that f is 1-connective. Then for E any truncating invariant, $E(R^{h\mathbb{Z}}) \to E(S^{h\mathbb{Z}})$ is an equivalence. Moreover, if f is n-connective, then $\mathrm{TC}(R^{h\mathbb{Z}}) \to \mathrm{TC}(S^{h\mathbb{Z}})$ is too.

We also obtain the following variant:

Theorem C (Levy)**.** Let $R \to S$ be a 1-connective map of (-1)-connective rings such that $\pi_{-1}R$ is a finitely generated $\pi_0 R$-module. Then for E any truncating invariant, $E(R) \to E(S)$ is an equivalence.

2. Reducing Theorem B to an easier problem

Let us first review a couple of definitions used in the above theorem statements, and then our first goal will be to come up with a strategy for proving Theorem B.

Definition 2.1 (localizing invariant)**.** $E : \mathrm{Cat}^{\mathrm{Perf}} \to \mathcal{D}$ is a *localizing invariant* if it commutes with filtered colimits, inverts Morita equivalences, and sends an exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$ to a fibre sequence in $\mathcal{D}$. Here $\mathcal{D}$ is stable.

Definition 2.2 (truncating invariant)**.** A localizing invariant E is *truncating* if for any connective $\mathbb{E}_1$-ring spectrum A, we have $E(A) \cong E(\pi_0 A)$ via the canonical map.

Example 2.3. K-theory, TC, THH, etc. are localizing invariants; the fibre of the cyclotomic trace, $K^{\mathrm{inv}} := \mathrm{fib}(\mathrm{tr}_{\mathrm{cyc}} : K \to \mathrm{TC})$, is a truncating invariant (by Dundas-Goodwillie-McCarthy applied to the map $A \to \pi_0 A$). K^{inv} is going to be the key example in this talk.

For completeness, we state here two interesting results derived from [LT19] which will not be used in the rest of the talk:

Theorem 2.4 ([LT19] nilinvariant)**.** *A truncating invariant E is nilinvariant: for every nilpotent two-sided ideal $I \subseteq A$ in a discrete unital ring A, $E(A) \simeq E(A/I)$.*

Theorem 2.5 ([LT19])**.** *Suppose there exists $n \geqslant 0$ for a localizing invariant E such that $E(R) \simeq E(\tau_{\leqslant n} R)$ for any connective $\mathbb{E}_1$-ring R. Then E is a truncating invariant.*

We would now like to approach the problem of proving Theorem B. We note that a $\mathbb{Z}$-action is determined by an automorphism η. So we have the following pullback diagram:

$$\begin{array}{ccc} R^{h\mathbb{Z}} & \longrightarrow & R \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (1,\eta)} \\ R & \xrightarrow{\Delta} & R \times R. \end{array}$$

Since $R^{h\mathbb{Z}}$ is an extension of $R \times R[-1]$ by $R \times R$ and R is connective, $R^{h\mathbb{Z}}$ is (-1)-connective. Thinking of $S^{h\mathbb{Z}}$ in terms of a similar pullback, we obtain a map of diagrams between the pullback square for $R^{h\mathbb{Z}}$ and that of $S^{h\mathbb{Z}}$, from which we may deduce that $f^{h\mathbb{Z}} : R^{h\mathbb{Z}} \to S^{h\mathbb{Z}}$ is 0-connective.

Question 2.6. If we apply a localizing invariant E to a pullback, is the resulting diagram still a pullback?

Answer 2.7. No, but there is a method for modifying the original square so it becomes pullback under E, given by the main theorem of [LT19].

Theorem 2.8 (Land-Tamme[LT19])**.** *Consider* $\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D \end{array}$ *a pullback square of $\mathbb{E}_1$-ring spectra. Associated to this square there exists a natural $\mathbb{E}_1$-ring spectrum $B \odot_A^D C$ with the following properties: The original diagram extends to a commutative diagram of $\mathbb{E}_1$-ring spectra*

$$\begin{array}{ccccc} A & \longrightarrow & C & & \\ \downarrow & & \downarrow & \searrow & \\ B & \longrightarrow & B \odot_A^D C & & \\ & \searrow & & \searrow & \\ & & & & D \end{array}$$

and any localizing invariant sends the inner square to a pullback square.

The underlying spectrum of $B \odot_A^D C$ is equivalent to $B \otimes_A C$, and the underlying diagram of spectra is the canonical one.

Remark 2.9. We may denote $B \odot_A^D C$ as $\odot$, perhaps with single subscript, when the originating pullback square is clear.

We will address the proof of Theorem 2.8 in Section 5 of the talk, but for now let's use it to reduce Theorem B to an easier problem. By Theorem 2.8, we have a map of pullback squares:

$$\begin{array}{ccc} E(R^{h\mathbb{Z}}) & \longrightarrow & E(R) \\ \downarrow & \lrcorner & \downarrow \\ E(R) & \longrightarrow & E(\odot_R) \end{array} \quad \overset{\text{“}E(f)\text{”}}{\curvearrowright} \quad \begin{array}{ccc} E(S^{h\mathbb{Z}}) & \longrightarrow & E(S) \\ \downarrow & \lrcorner & \downarrow \\ E(S) & \longrightarrow & E(\odot_S) \end{array}$$

where $\odot_R = R \overset{R\times R}{\underset{R^{h\mathbb{Z}}}{\odot}} R$. We want to show that $E(R^{h\mathbb{Z}}) \cong E(S^{h\mathbb{Z}})$, so it is enough to show that $E(R) \cong E(S)$ and $E(\odot_R) \cong E(\odot_S)$.

Proposition 2.10. *In the setting of theorem B, $E(f) : E(R) \to E(S)$ is an equivalence.*

Proof. Consider the diagram

$$\begin{array}{ccc} E(R) & \xrightarrow{E(f)} & E(S) \\ \downarrow \simeq & & \downarrow \simeq \\ E(\pi_0 R) & \underset{\simeq}{\xrightarrow{E(\pi_0 f)}} & E(\pi_0 S). \end{array}$$

Vertical arrows are equivalences from the definition of a truncating invariant, and the bottom horizontal arrow is an equivalence since f is 1-connective. Thus, $E(f)$ is also an equivalence. □

We are going to check $\odot_R$ and $\odot_S$ are connective and the induced map $\tilde{f} : \odot_R \to \odot_S$ is 1-connective. Then the same argument as Proposition 2.10 shows that $E(\tilde{f})$ is an equivalence. This can all be done on the level of underlying spectra so we can work with $f \otimes f : R \otimes_{R^{h\mathbb{Z}}} R \to S \otimes_{S^{h\mathbb{Z}}} S$ directly and not worry about what $\odot$ really is. It also follows that $\mathrm{TC}(\tilde{f})$ is 2-connective.

3. Connectivity results and proof of Theorem B

Recall that we have a standard t-structure on Mod_A for some $\mathbb{E}_1$-ring A: the connective modules are objects generated under colimits and extensions by A (see [Lur17] Proposition 1.4.4.11). The coconnected/coconnective objects are those with coconnected/coconnective underlying spectra.

Lemma 1. (Levy) Let A be a (-1)-connective $\mathbb{E}_1$-ring spectrum, M an A-module with underlying spectrum connective. Then M is connective in the standard t-structure on A-modules. In particular, if N is a right A-module with connective underlying spectrum then $N \otimes_A M$ is connective.

Proof. For an A-module M with connective underlying spectrum, we want to show

$$M \in A\text{-Mod}_{\geqslant 0} := \tau_{\geqslant 0}(A\text{-Mod}).$$

Decompose as $\tau_{\geqslant 0}M \to M \to \tau_{<0}M$.

The underlying spectrum of A is (-1)-connective, so $\tau_{\geqslant 0}M$ is too since it's built from A via colimits and extensions. M has a connective underlying spectrum by assumption. Hence after rotating the triangle, $\tau_{<0}M$ sits as an extension of connective spectra so is itself connective. But it's also coconnected by assumption, so it must be zero. Hence $M = \tau_{\geqslant 0}M$ is in the connective part of the t-structure.

Now, $M \otimes_A N$ is built from $A \otimes_A N = N$ out of colimits and extensions, and N is connective. Then $M \otimes_A N$ is connective. □

Question 3.1. Which part of what we wanted to prove does this lemma address?

Answer 3.2. It shows $R \otimes_{R^{h\mathbb{Z}}} R$ and $S \otimes_{S^{h\mathbb{Z}}} S$ are connective, since R, S are connective by assumption and $R^{h\mathbb{Z}}, S^{h\mathbb{Z}}$ are (-1)-connective.

It remains to address the connectivity of the map $f \otimes f : R \otimes_{R^{h\mathbb{Z}}} R \to S \otimes_{S^{h\mathbb{Z}}} S$.

Lemma 2. (Levy) Suppose $A \to A'$ is an i-connective map of (-1)-connective $\mathbb{E}_1$-rings, $i \geqslant -1$. Let M, N respectively be right and left A'- modules, then $M \otimes_A N \to M \otimes_{A'} N$ is $(i+1)$-connective.

We're going to finish the proof of Theorem B by factoring the map $f \otimes f$ into several parts and addressing each separately. One of pieces is $S \otimes_{R^{h\mathbb{Z}}} S \to S \otimes_{S^{h\mathbb{Z}}} S$, and we will use Lemma 2 to establish 1-connectivity of this piece.

Proof. M, N built from A' under colimits and extensions. Thus, without loss of generality, we may assume that $M = N = A'$. We then need only consider the connectivity of the multiplication map $\mu : A' \otimes_A A' \to A' \otimes_{A'} A' = A' = A \otimes_A A'$.

It admits a section s induced by the left unit $A \to A'$, so $\mathrm{fib}(\mu) = \mathrm{cofib}(s)$. Connectivity of μ is connectivity of $\mathrm{fib}(\mu)$ by definition.

Let $X := \mathrm{cofib}(A \to A')$, which is $(i+1)$-connective by assumption. Then $\mathrm{cofib}(s) = X \otimes_A A'$ is an extension of $X \otimes_A A = X$ by $X \otimes_A X$. Since $X \otimes_A X = \Omega^{-2i-2}(\Omega^{i+1}X \otimes_A \Omega^{i+1}X)$ is $(2i+2)$-connective by Lemma 1, we see that $\mathrm{cofib}(s)$ is at least $(i+1)$-connective and $\mathrm{fib}(\mu)$ is too. Thus μ is $(i+1)$-connective. □

Now we are ready to finish the proof of Theorem B.

Proof of Theorem B. It remains only to show $R \otimes_{R^{h\mathbb{Z}}} R \to S \otimes_{S^{h\mathbb{Z}}} S$ is 1-connective. Factor as

$$\begin{array}{ccc} R \otimes_{R^{h\mathbb{Z}}} S & \longrightarrow & S \otimes_{R^{h\mathbb{Z}}} S \\ \uparrow & & \downarrow \\ R \otimes_{R^{h\mathbb{Z}}} R & \xrightarrow{f \otimes f} & S \otimes_{S^{h\mathbb{Z}}} S. \end{array}$$

Notice that $f^{h\mathbb{Z}}$ is 0-connective, so we can use Lemma 2 with $i = 0$ to see that the right vertical map is 1-connective. Since $R^{h\mathbb{Z}}$ is (-1)-connective, Lemma 1 tells us that the other two component maps are 1-connective because their fibres are. Thus the composite $f \otimes f$ is 1-connective. □

4. Computation of $K(L_{K(1)}\mathbb{S})$ and proof of Theorem A

We will now see some applications of Theorem B. We think of this result as a replacement for Dundas-Goodwillie-McCarthy in situations where we are dealing with (-1)-connective rings. Recall that $K^{\mathrm{inv}} := \mathrm{fib}(\mathrm{tr}_{\mathrm{cyc}} : K \to \mathrm{TC})$ is truncating invariant. This is an easy consequence of DGM, applied to the map $R \to \pi_0 R$ for a connective $\mathbb{E}_1$-ring R. Let's use theorem B and see what this buys us.

Corollary 4.1. *Let R be a connective $\mathbb{E}_1$-ring with $\mathbb{Z}$-action. $L_{T(n)}K(R^{h\mathbb{Z}}) \xrightarrow{\simeq} L_{T(n)}\,\mathrm{TC}(R^{h\mathbb{Z}})$, $n \geqslant 2$.*

Proof. $\pi : R^{h\mathbb{Z}} \to \pi_0 R$ is a 1-connective map of $\mathbb{E}_1$-rings. By Theorem B and the fact that K^{inv} is a truncating invariant, $K^{\mathrm{inv}}(\pi) : K^{\mathrm{inv}}(R^{h\mathbb{Z}}) \xrightarrow{\simeq} K^{\mathrm{inv}}((\pi_0 R)^{h\mathbb{Z}})$ is an equivalence.

Claim. After $T(n)$-localizing, RHS vanishes, that is, $L_{T(n)}K^{\mathrm{inv}}((\pi_0 R)^{h\mathbb{Z}}) \simeq 0$.

Hence LHS also vanishes, so $L_{T(n)}K(R^{h\mathbb{Z}}) \xrightarrow{\simeq} L_{T(n)}\,\mathrm{TC}(R^{h\mathbb{Z}})$ as desired. □

Proof of the claim. We must show that $K((\pi_0 R)^{h\mathbb{Z}}) \to \mathrm{TC}((\pi_0 R)^{h\mathbb{Z}})$ is a $T(n)$-local equivalence. In fact, the domain and codomain both vanish $T(n)$-locally for $n \geq 2$ because they each have height ≤ 1. Note $\pi_0 R$ has height ≤ 0, and $\pi_0 R^{h\mathbb{Z}}$ has finitely many nontrivial homotopy groups, so is bounded above and thus also height 0. By purity results, K and TC shift height up by at most 1, so $K((\pi_0 R)^{h\mathbb{Z}})$ and $\mathrm{TC}((\pi_0 R)^{h\mathbb{Z}})$ have height ≤ 1. □

Remark 4.2. In last week's talk, Maxime Ramzi outlined a proof of a similar result when $R^{h\mathbb{Z}}$ is replaced by something connective, using [DGM12]. But $R^{h\mathbb{Z}}$ is not connective so we have really achieved something new.

We now wish to prove Theorem A for $p > 2$. Recall that ℓ is the (p-complete) Adams summand of connective topological K-theory, with $\mathbb{Z}$-action via the Adams operation ψ^{p+1}. The underlying spectrum of $\ell^{h\mathbb{Z}}$ is the (-1)-connective cover of the $K(1)$-local sphere, and $\ell^{h\mathbb{Z}}$ is an $\mathbb{E}_\infty$-ring. We will need the following technical ingredient for the proof of Theorem A:

Theorem 4.3 (Devissage, [BL21])**.** *If R is a coconnective ring with π_0 regular, and π_{-i} has tor dimension $< i$ over π_0, then the connective cover map $\pi_0 R \to R$ is an equivalence on K-theory.*

Proof of Theorem A ($p > 2$). The pullback square is an easy consequence of Theorem B. Since $\ell \to \mathbb{Z}_p$ is a 1-connective map of connective $\mathbb{E}_1$-rings with $\mathbb{Z}$ action, $K^{\mathrm{inv}}(\ell^{h\mathbb{Z}}) \simeq K^{\mathrm{inv}}(\mathbb{Z}_p^{h\mathbb{Z}})$ by Theorem B. Noting that $K(\mathbb{Z}_p^{h\mathbb{Z}}) \cong K(\mathbb{Z}_p)$, we get the pullback square. Theorem B also gives the desired connectivity of F.

For the cofibre sequence, we start with the localization sequence $\mathrm{Sp}_{\geqslant n+1} \to \mathrm{Sp} \to L_n^f\,\mathrm{Sp}$. Fix $n = 1$ and tensor over Sp^ω with $\mathrm{Perf}(\ell^{h\mathbb{Z}})$. We get

$$\mathrm{Sp}^\omega_{\geqslant 2} \otimes \mathrm{Perf}(\ell^{h\mathbb{Z}}) \to \mathrm{Perf}(\ell^{h\mathbb{Z}}) \to \mathrm{Perf}(L_1^f \ell^{h\mathbb{Z}}).$$

Now, $L_1^f(\ell^{h\mathbb{Z}}) \simeq L_{K(1)}\mathbb{S}$, so after taking K-theory, we can identify the third term with $K(L_{K(1)}\mathbb{S})$. The middle term yields $K(\ell^{h\mathbb{Z}})$. It remains to identify $K(\mathrm{Sp}^\omega_{\geqslant 2} \otimes \mathrm{Perf}(\ell^{h\mathbb{Z}}))$ with $K(\mathbb{F}_p)$.

By the thick subcategory theorem, any type 2 spectrum Z generates $\mathrm{Sp}^\omega_{\geqslant 2}$, and then $Z \otimes \ell^{h\mathbb{Z}}$ automatically generates $\mathrm{Sp}^\omega_{\geqslant 2} \otimes \mathrm{Perf}(\ell^{h\mathbb{Z}})$. Choose Z so that $Z \otimes \ell = \mathbb{F}_p$, e.g. taking $Z = \mathbb{S}/(p, v_1)$ to be the Smith-Toda complex works. Then $Z \otimes \ell^{h\mathbb{Z}} = \mathbb{F}_p^{h\mathbb{Z}}$ is coconnective with $\pi_0 = \mathbb{F}_p$. To compute $K(\mathrm{Sp}^\omega_{\geqslant 2} \otimes \ell^{h\mathbb{Z}})$, we note that by Morita theory, $K(\mathrm{Sp}^\omega_{\geqslant 2} \otimes \ell^{h\mathbb{Z}}) \cong K(\mathrm{End}(Z \otimes \ell^{h\mathbb{Z}}))$. This endomorphism ring splits as a tensor product,

$$\mathrm{End}(Z \otimes \ell^{h\mathbb{Z}}) \cong \mathrm{End}_{\mathrm{Sp}_{\geqslant 2}}(Z) \otimes \mathrm{End}_{\ell^{h\mathbb{Z}}}(\ell^{h\mathbb{Z}}) = Z \otimes Z^\vee \otimes \ell^{h\mathbb{Z}} = Z^\vee \otimes \mathbb{F}_p^{h\mathbb{Z}}.$$

Now $Z \otimes Z^\vee \otimes \ell^{h\mathbb{Z}}$ is still coconnective with $\pi_0 = \mathbb{F}_p$, so by Theorem 4.3 its K-theory is the K-theory of $\mathbb{F}_p$. Putting everything together,

$$K(\mathrm{Sp}^{\omega}_{\geqslant 2} \otimes \ell^{h\mathbb{Z}}) \cong K(Z \otimes Z^\vee \otimes \ell^{h\mathbb{Z}}) \cong K(\mathbb{F}_p).$$

□

5. Land-Tamme construction of ⊙

This section follows [LT19]. Recall that we are considering a pullback square of $\mathbb{E}_1$-ring spectra

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D \end{array},$$

and trying to replace D by some other $\mathbb{E}_1$-ring with a comparison map to D, so that the new square becomes a pullback upon applying any localizing invariant E. We want to fill in the diagram

$$\begin{array}{ccccc} E(A) & \longrightarrow & E(C) & & \\ \downarrow & \lrcorner & \downarrow & \searrow & \\ E(B) & \longrightarrow & E(B \odot_A^D C) & & \\ & \searrow & & \searrow & \\ & & & & E(D) \end{array}.$$

Definition 5.1 (Lax pullback)**.** Given a diagram of (stable) ∞-categories $\mathcal{B} \xrightarrow{p} \mathcal{D} \xleftarrow{q} \mathcal{C}$, the corresponding lax pullback $\mathcal{B} \vec{\times}_{\mathcal{D}} \mathcal{C}$ is the (stable) ∞-category defined by

$$\begin{array}{ccc} \mathcal{B} \vec{\times}_{\mathcal{D}} \mathcal{C} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{D}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\text{source, target})} \\ \mathcal{B} \times \mathcal{C} & \longrightarrow & \mathcal{D} \times \mathcal{D} \end{array}.$$

To be explicit, $\mathcal{B} \vec{\times}_{\mathcal{D}} \mathcal{C}$ has

- Objects: $(b \in \mathcal{B}, c \in \mathcal{C}, f : p(b) \to q(c))$.
- Mapping spaces:

$$\begin{array}{ccc} \mathrm{Map}\Big((b,c,f),(b',c',f')\Big) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(c,c') \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f^* q} \\ \mathrm{Map}_{\mathcal{B}}(b,b') & \xrightarrow{f'_* p} & \mathrm{Map}_{\mathcal{D}}(p(b), q(c')) \end{array}$$

from understanding a map g in $\mathcal{B} \vec{\times}_{\mathcal{D}} \mathcal{C}$ as a homotopy-commutative diagram

$$\begin{array}{ccc} p(b) & \xrightarrow{f} & q(c) \\ {\scriptstyle p(g_{\mathcal{B}})}\downarrow & & \downarrow{\scriptstyle q(g_{\mathcal{C}})} \\ p(b') & \xrightarrow{f'} & q(c') \end{array}.$$

Remark 5.2. This is manifestly a functorial construction. Here are some more facts about the lax pullback construction which we won't prove, but which can be found in [Tam18].

(1) The arrow in the notation for the lax pullback denotes the artificial orientation we have introduced between $\mathcal{B}$ and $\mathcal{C}$. In particular, inside the lax pullback there is one-directional hom-vanishing from $\mathcal{C}$ to $\mathcal{B}$, $\mathrm{Map}((0,c,0),(b,0,0)) = 0$, but not vice versa.

(2) Suppose $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are stable, $\mathcal{B}$ and $\mathcal{C}$ are compactly generated presentable, $\mathcal{D}$ is cocomplete, p and q are exact and preserve small colimits, and p preserves compact objects. Then $\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$ is compactly generated presentable with $(\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C})^\omega = \mathcal{B}^\omega\vec{\times}_{\mathcal{D}}\mathcal{C}^\omega$.
(3) Every object (x, y, f) of $\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$ sits in a fibre sequence $(0, y, 0) \to (x, y, f) \to (x, 0, 0)$. As a consequence, any set of generators of $\mathcal{B}$ together with generators of $\mathcal{C}$ will generate $\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$. This also works for compact generators.

Lemma 5.3. *For E any localizing invariant, $E(\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}) \simeq E(\mathcal{B}) \bigoplus E(\mathcal{C})$.*

Proof. It is easy to check using the formula for mapping spaces that $\mathcal{B} \overset{\text{f.f.}}{\hookrightarrow} \mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$, $\mathcal{C} \overset{\text{f.f.}}{\hookrightarrow} \mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$ and $\mathrm{Map}_{\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}}(\mathcal{C}, \mathcal{B}) = 0$. Then we get a split exact sequence

$$\mathcal{C} \underset{j_{\mathcal{C}}}{\overset{\mathrm{pr}_{\mathcal{C}}}{\rightleftarrows}} \mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C} \underset{\mathrm{pr}_{\mathcal{B}}}{\overset{j_{\mathcal{B}}}{\rightleftarrows}} \mathcal{B}$$

with pr denoting projection and j the fully faithful inclusions. It is easy to check the claimed adjunctions by computing the relevant mapping spaces as pullbacks; note that the two adjunctions aren't symmetric in $\mathcal{B}$ and $\mathcal{C}$. This split exact sequence exhibits $\mathcal{B}$ as the localization of $\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}$ at $\mathcal{C}$. Since E is localizing invariant, $E(\mathcal{B}\vec{\times}_{\mathcal{D}}\mathcal{C}) \simeq E(\mathcal{C}) \bigoplus E(\mathcal{B})$ as desired. □

How are we going to use the lax pullback construction? Let's consider the category $\mathcal{M} = \mathrm{Perf}(B)\vec{\times}_{\mathrm{Perf}(D)}\mathrm{Perf}(C)$. Then $\mathcal{M}$ is stable, presentable, and compactly generated by the two objects $(B, 0, 0)$ and $(0, C, 0)$.

Lemma 5.4. *$i_A : \mathrm{Perf}(A) \hookrightarrow \mathcal{M} = \mathrm{Perf}(B)\vec{\times}_{\mathrm{Perf}(D)}\mathrm{Perf}(C)$ with $A \mapsto (B, C, id_D)$ is a fully faithful embedding.*

Proof. $\mathrm{Perf}(A)$ is generated by $A = B \times_D C$. We check only a single mapping space in $\mathcal{M}$,

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{M}}(i_A(A), i_A(A)) & \longrightarrow & \mathrm{Map}_C(C, C) = C \\ \downarrow & \lrcorner & \downarrow \\ B = \mathrm{Map}_B(B, B) & \longrightarrow & \mathrm{Map}_D(D, D) = D \end{array}$$

but this is the same diagram as the original pullback square, so the pullback is $A = \mathrm{Map}_A(A, A)$. □

Let $\mathcal{Q} := \mathrm{cofib}(\mathrm{Perf}(A) \hookrightarrow \mathrm{Perf}(B)\vec{\times}_{\mathrm{Perf}(D)}\mathrm{Perf}(C))$ and $\pi : \mathcal{M} \to \mathcal{Q}$ the corresponding map.

Lemma 5.5. *$\mathcal{Q}$ has a single generator, $\pi(B, 0, 0) \sim \pi(0, C, 0)$*

Proof. Since $\mathcal{M}$ is generated by $(B, 0, 0)$ and $(0, C, 0)$, we know that the images of these under π together generated $\mathcal{Q}$. Consider the fibre sequence $(0, C, 0) \to (B, C, \mathrm{id}_D) = i_A(A) \to (B, 0, 0)$ in $B\vec{\times}_D C$. Then $\pi(0, C, 0) = \Omega\pi(B, 0, 0)$ in $\mathcal{Q}$, either one (compactly) generates $\mathcal{Q}$. □

Definition 5.6. We can now define $B \odot_A^D C = \mathrm{End}_{\mathcal{Q}}(\pi(0, C, 0))$. Since $\mathcal{Q}$ is stable, it is enriched in spectra and so the $\odot$ construction naturally has the structure of an $\mathbb{E}_1$-ring.

We are now ready to prove Theorem 2.8, i.e. that $B \odot_A^D C$ satisfies all of the properties claimed.

Proof of Theorem 2.8. Since $\mathcal{Q}$ is cocomplete and stable with $B \odot_A^D C$ the endomorphism ring of a compact generator, by Schwede-Shipley we have $\mathcal{Q} \simeq \mathrm{Perf}(B \odot_A^D C)$. This gives us a localization sequence

$$\mathrm{Perf}(A) \xrightarrow{i_A} \mathrm{Perf}(B)\vec{\times}_{\mathrm{Perf}(D)}\mathrm{Perf}(C) \xrightarrow{\pi} \mathrm{Perf}(B \odot_A^D C). \tag{1}$$

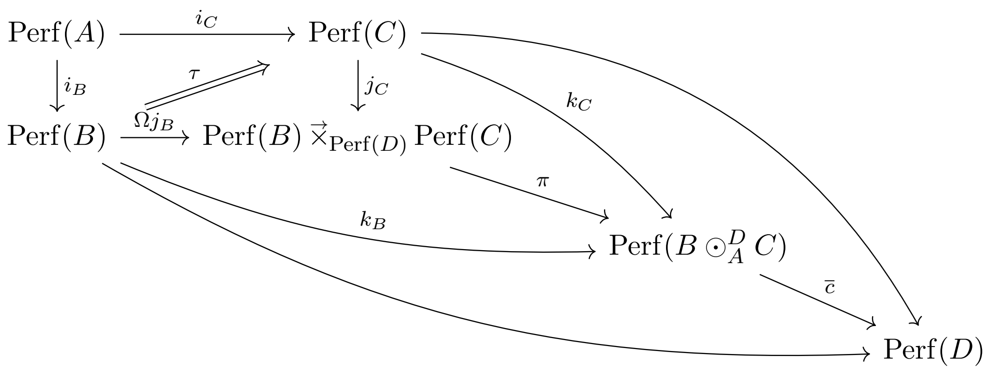
Applying any localizing invariant E yields a fibre sequence

$$\begin{array}{ccccc} E(\mathrm{Perf}(A)) & \longrightarrow & E\left(\mathrm{Perf}(B) \vec{\times}_{\mathrm{Perf}(D)} \mathrm{Perf}(C)\right) & \longrightarrow & E(\mathrm{Perf}(B \odot_A^D C)) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ E(A) & \xrightarrow{(i_B, i_C)} & E(B) \bigoplus E(C) & \xrightarrow{\pi j_B + \pi j_C} & E(B \odot_A^D C) \end{array}$$

This being a fibre sequence tells us that the square

$$\begin{array}{ccc} E(A) & \xrightarrow{i_C} & E(C) \\ \downarrow i_B & \lrcorner & \downarrow \pi j_C \\ E(B) & \xrightarrow{\pi j_B} & E(B \odot_A^D C) \end{array}$$

is a pullback. Next we want to fill out the diagram



.

We proceed as follows:

- k_B, k_C are defined as compositions so the two triangles commute;
- the natural transformation $\tau : \Omega j_B i_B \to j_C i_C$ is induced by the fibre sequence $(0, y, 0) \to (x, y, f) \to (x, 0, 0)$ for any object of the lax pullback;
- $\pi(\tau)$ is a natural equivalence (between two copies of $\mathrm{Perf}(A)$ inside $\mathcal{Q}$) so the outer diagram commutes;
- maps respect the preferred generators since $k_B i_B(A) \simeq \pi(0, C, 0) \simeq k_C i_C(A)$ via $\pi(\tau)$;
- the map $\mathrm{Perf}(B) \vec{\times}_{\mathrm{Perf}(D)} \mathrm{Perf}(C) \xrightarrow{c} \mathrm{Perf}(D)$ is obtained by taking the cofibre of the map f in each object (x, y, f) of the lax pullback. This factors through $\mathcal{Q}$, inducing the map $\bar{c}$ which sends the generator $\pi(0, C, 0)$ of $\mathcal{Q}$ to D.

One can check that the outer square obtained by pasting all the natural transformations together is the same as that in the original pullback square of rings. Passing to the endomorphism ring of our preferred generators everywhere fills out the diagram 2.8 we wanted.

Finally, we claimed that the underlying spectrum of $B \odot_A^D C$ is equivalent to $B \otimes_A C$. This is a consequence of the following lemma below, from [LT19].

Lemma 5.7 ([LT19])**.** *The Ind-completion of the localizing sequence (1) is split*

$$\begin{array}{ccccc} \mathrm{Ind}(\mathrm{Perf}(A)) & \underset{i_A}{\overset{s_A}{\rightleftarrows}} & \mathrm{Ind}(\mathrm{Perf}(B) \vec{\times}_{\mathrm{Perf}(D)} \mathrm{Perf}(C)) & \underset{\pi}{\overset{r}{\leftrightarrows}} & \mathrm{Ind}(\mathcal{Q}) \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ \mathrm{RMod}(A) & \leftrightarrows & \mathrm{RMod}(B) \vec{\times}_{\mathrm{RMod}\,(D)} \mathrm{RMod}(C) & \leftrightarrows & \mathrm{RMod}(B) \odot_{\mathrm{RMod}(D)} \mathrm{RMod}(C) \end{array}$$

and π is a Bousfield localization, i.e. it admits a fully faithful right adjoint r. The localization functor $L = r\pi$ is given by the cofibre $\mathrm{cofib}(i_A s_A(-) \to (-))$ *of the counit transformation of the adjunction (i_A, s_A).*

By the lemma, we get a cofibre sequence of C-bimodules

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{M}}(j_C(C), i_A s_A j_C(C)) & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(j_C(C), j_C(C)) & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(j_C(C), Lj_C(C)) \\ & & \simeq\uparrow & & \simeq\uparrow \\ & & C \simeq \mathrm{End}_C(C) & \longrightarrow & \mathrm{End}_{\mathcal{M}}(Lj_C(C)) \simeq B \odot_A^D C \end{array} .$$

Let us denote $I := i_A s_A j_B(B)$, and then the above digram becomes the cofibre sequence

$$\begin{array}{ccccc} I \otimes_A C & \longrightarrow & C & \longrightarrow & B \otimes_A C \\ \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\ I \otimes_A C & \longrightarrow & C & \longrightarrow & B \odot_A^D C \end{array} .$$

Since $B \odot_A^D C$ and $B \otimes_A C$ are cofibres of the same map of C-bimodules, they agree on the level of underlying spectra. □

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