K THEORY FROM THE GROUND UP – TALK NOTES

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Contents

Outline

(1) classical constructions (of K_0, K_1, K_2)
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These notes were prepared for a talk I gave at the Zygotop seminar at Harvard. I wrote them for myself and do not guarantee that they are free of errors! You have been warned. Much of the material is drawn from Weibel's book [1]. I also found the expository notes of Brazelton [2] helpful. The material on universal constructions of connective and non-connective K-theory is from [3].

Introduction

Fundamentally, K-theory is an invariant which takes in some input – e.g. a ring, a stable ∞-category, a symmetric monoidal category, a Waldhausen category, etc – and outputs a ring spectrum. The classical constructions of K groups can be recovered as the (stable) homotopy groups of this spectrum. For connective K-theory, all the negative homotopy groups vanish, and non-connective K-theory is a spectrum whose 0-truncation recovers connective K-theory, but where the negative homotopy groups no longer need vanish.

In this talk, we'll start with classical constructions of K_0 in several different settings, and also of K_1 and K_2 for a ring. Then we'll discuss 5 different constructions of higher K-theory: in terms of the universal localising invariant, the $+$, Q, and S constructions, and as an ∞-group completion. Some of these are more epistemologically satisfying than others (the universal localising invariant perspective), some are more correct than others (group completion is usually wrong), and some are more computable than others (mostly the $+$ construction); most will satisfy some notion of Morita equivalence. We'll explain how nearly all of these can be used to compute connective K-theory of a ring, and how to extend to non-connective K-theory.

Then we'll compute $K(\mathrm{FinSet}) = \mathbb{S}$ and $K(\mathbb{F}_q)$, and we will survey some recent results and general properties, time permitting.

1. Classical constructions

1.1. K_0 of a ring.

K_0 is supposed to be the group completion of the monoid of vector bundles on a topological space. For a ring, we can mimic this as

Definition 1.1. For a ring R, define $K_0(R)$ to be the group completion of the abelian monoid of finitely generated projective modules on R (under direct sum).

When R is commutative, $K_0(R)$ is a commutative ring whose product operation is $\otimes_R$. You can check for yourself that this is well-defined on the group completion and distributes over addition.

Proposition 1.2. *If R and S are Morita equivalent (i.e. their module categories are additively equivalent) then $K_0(R) \cong K_0(S)$.*

Some facts: $K_0(F) = \mathbb{Z}$ for F a field. More generally, if there exists a map $R \to F$ for some field F then $\mathbb{Z}$ is a direct summand of $K_0(R)$.

1.2. K_0 of a Waldhausen category.

First, what is a Waldhausen category?

Definition 1.3. A *Waldhausen category* is a category $\mathcal{C}$ with two distinguished classes of morphisms, called cofibrations and weak equivalences, satisfying:

(a) $\mathcal{C}$ has a zero object and the unique map $0 \to A$ is always a cofibration;
(b) any isomorphism is both a cofibration and a weak equivalence;
(c) weak equivalences and cofibrations are both closed under composition;
(d) the pushout of a cofibration along any morphism exists and is a cofibration;
(e) and glueing for weak equivalences.

Note that the conditions on cofibres mean that all coproducts exist in $\mathcal{C}$ (pushout of the zero morphism) and any cofibration has a cokernel (also a pushout).

Example 1.4. Any category with cofibrations can be made Waldhausen by taking the weak equivalences to just be isomorphisms. e.g. If $\mathcal{C}$ is an exact category (additive subcategory of an abelian category closed under extensions) then it can be considered as a Waldhausen category by taking the cofibration sequences to be short exact sequences (and cofibrations to be those monomorphisms appearing in an SES, i.e. whose cofibres lie in the subcategory).

Given a Waldhausen category, we define K_0 by taking the objects up to weak equivalence as (commuting) generators, and cofibre sequences as relations.

Definition 1.5. For a Waldhausen category $\mathcal{C}$, the abelian group K_0 has a presentation with generators $[C]$ for $C \in \operatorname{ob} \mathcal{C}$, and relations:

- $[C] = [C']$ given a weak equivalence $C \xrightarrow{\sim} C'$,
- $[B] = [A] + [B/A]$ for a cofibration sequence $A \hookrightarrow B \twoheadrightarrow B/A$.

Remark 1.6. Thinking of an abelian category as a Waldhausen category as above recovers the usual notion of K_0.

1.3. **K_0 of a symmetric monoidal category.** This works by group completion, like for a ring.

Remark 1.7. Any category with finite products (or finite coproducts) is symmetric monoidal, taking this to be the tensor product.

Definition 1.8. Given a symmetric monoidal category S, suppose we have a set of isomorphism classes of objects. Then this is an abelian monoid under the tensor product; its group completion is $K_0(S)$.

Example 1.9. FinSet is symmetric monoidal under disjoint union. Then $K_0(\text{FinSet}) = \mathbb{Z}$ (the monoid is $(\mathbb{Z}_{\geq 0}, +)$ classified by number of elements).

1.4. **K_1 and K_2 of a ring.** First, let's define $K_1(R)$ two (slightly) different ways. We have embeddings

$$\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \dots$$

via taking a block matrix adding a 1 on the diagonal. The resulting union is the infinite general linear group, $\mathrm{GL}(R)$.

Definition 1.10. For a ring R, take

$$K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$$

to be the abelianization of the infinite general linear group.

Note that K_1 is functorial. One can also show that it is Morita invariant.

Lemma 1.11. *The commutator subgroup of* $\mathrm{GL}(R)$ *is generated by elementary matrices, that is matrices where all but one (off-diagonal) entry agrees with an identity matrix* I_n*.*

Proof. Easy to check that every elementary matrix lies in the commutator subgroup. Conversely, each commutator in $\mathrm{GL}_n(R)$ can be written as a product of elementary matrices in $\mathrm{GL}_{2n}(R)$. □

Note that diagonal matrices aren't in the commutator subgroup because they already commute with everything, so these aren't considered elementary.

Example 1.12. For a Euclidean domain R, we have $K_1(R) = R^\times$, with the isomorphism via the determinant map. In particular this holds when R is a field or $R = \mathbb{Z}$ or $R = F[t]$ with F a field.

Now let us address K_2. One may define the Steinberg groups $\mathrm{St}_n(R)$ in terms of generators and relations, with generators $x_{ij}(r)$ for $1 \leq i, j \leq n$ and $i \neq j$, and $r \in R$. The relations are satisfied by the elementary matrices, so letting $\mathrm{St}(R)$ be the union of the $\mathrm{St}_n(R)$ we obtain a map $\mathrm{St}(R) \to \mathrm{GL}(R)$ whose image is the (subgroup generated by) elementary matrices. Then define $K_2(R)$ so that the sequence

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 0$$

is exact.

Lemma 1.13. *K_2 is a functor from rings to abelian groups.*

Example 1.14. $K_2(\mathbb{Z})$ is cyclic of order 2, via the Euclidean algorithm. Similar arguments show that $K_2(F[t]) = K_2(F)$ for every field F. We will see later that in fact this holds for every regular ring.

Proposition 1.15. *For a field F, $K_2(F)$ is generated by Steinberg symbols $\{x, y\}$ with $x, y \in F^\times$, subject to bilinearity and the Steinberg relation $\{x, 1 - x\} = 1$ for $x \neq 0, 1$.*

That is, $K_2(F)$ is the quotient of $F^\times \otimes F^\times$ by elements $x \otimes (1 - x)$.

Corollary 1.16. *$K_2(\mathbb{F}_q) = 1$ for a finite field.*

1.5. **Negative K groups of a ring.** What we really want is to have, given a Verdier sequence $A \to B \to C$, an induced LES of K groups ($K(A) \to K(B) \to K(C)$)

$$\ldots \to K_i(A) \to K_i(B) \to K_i(C) \to K_{i-1}(A) \to \ldots$$

but to make this work, we need non-connective K-theory in many cases. (Alternatively: we want K-theory to satisfy excision so it's a cohomology theory.)

In particular, excision applies in the situation of gluing two $\mathbb{A}^1$s along $\mathbb{G}_m$ to form $\mathbb{P}^1$. We have a diagram like

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \qquad \begin{array}{ccc} R[t,t^{-1}] & \longleftarrow & R[t] \\ \uparrow & & \uparrow \\ R[t^{-1}] & \longleftarrow & R \end{array}$$

Via excision, you get a sequence

$$K(\mathbb{P}^1) \to K(\mathbb{A}^1) \oplus K(\mathbb{A}^1) \to K(\mathbb{G}_m)$$

which is supposed to induce an LES on homotopy groups. We can compute $K(\mathbb{P}^1) = K(R) \oplus K(R)$ and the map $* \to \mathbb{A}^1 \to *$ splits off a summand of $K(R)$ from $\mathbb{A}^1$. Call the remaining summand $NK(R)$. Then we have

$$K_i(R) \oplus K_i(R) \to K_i(\mathbb{A}^1) \oplus K_i(\mathbb{A}^1) \to K_i(\mathbb{G}_m) \to K_{i-1}(R) \oplus K_{i-1}(R)$$

which is supposed to be exact. The first map is diagonal because the two generators $\mathcal{O}, \mathcal{O}(1)$ for $K(R)$ restrict to the same thing on both copies of $\mathbb{A}^1$. So the only summand of $K_0(\mathbb{G}_m)$ not coming from the previous term is a single copy of $K_{-1}(R)$. In terms of rings, this means we can define

$$K_{-1}(R) = \operatorname{coker}(K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])).$$

This gives $K_0(\mathbb{G}_m) \cong K_0(R) \oplus K_{-1}(R) \oplus NK_0(R) \oplus NK_0(R)$.

This sequence is supposed to extend to further negative K-groups, which allows us to define as follows.

Definition 1.17. For $n > 0$, inductively define $K_{-n}(R)$ to be the cokernel of the natural map

$$K_{1-n}(R[t]) \oplus K_{1-n}(R[t^{-1}]) \to K_{1-n}(R[t,t^{-1}]).$$

This is constructed to extend the sequence for $\mathbb{P}^1, \mathbb{G}_m, \mathbb{A}^1$ to the right, to give a LES. (Still need to extend to the left by defining higher K-groups.)

2. Five constructions of higher K-theory

2.1. **Universal construction.** Here is a first definition of K-theory in modern language. This definition is extremely universal and makes many of the formal properties clear, but is difficult to compute with.

Definition 2.1. A functor $E : \mathrm{Cat}_\infty^{ex} \to \mathrm{Sp}$ is a localising invariant if

(1) E commutes with filtered colimits
(2) E is Morita invariant (i.e. sends idempotent completion $\mathscr{C} \to \mathscr{C}^{idem} = \mathrm{Ind}(\mathscr{C})^\omega$ to an isomorphism)
(3) E is exact i.e. given a sequence which is both fibre and cofibre, image is also a fibre/cofibre sequence

Note that Cat_∞^{ex} (category of small stable ∞-categories and exact functors, i.e. functors preserving finite limits and colimits) is not itself stable so we need to demand that in the domain the sequence is both fibre AND cofibre. Can also replace the codomain Sp with any stable presentable ∞-category.

An *additive invariant* is the same, but we only require that split exact sequences are sent to cofibre sequences, rather than all exact sequences.

Remark 2.2. What is a fibre and cofibre sequence in Cat_∞^{ex}? The first functor must be fully faithful, and the second functor identifies the third category as the Verdier quotient. This is usually easy to check.

There is a universal localising invariant

$$\mathrm{Cat}_\infty^{ex} \xrightarrow{U} \mathrm{NMot}$$

where NMot is called the category of noncommutative motives. Every localising invariant factors uniquely through this. Then $\mathrm{Sp}^\omega \in \mathrm{Cat}_\infty^{ex}$ so we can define

$$K(\mathscr{C}) = \mathrm{Map}_{\mathrm{NMot}}(U(\mathrm{Sp}^\omega), U(\mathscr{C})).$$

Since NMot is stable presentable, it is enriched in spectra, so this is a spectrum. The classical K-theory groups can be recovered as its homotopy groups.

Remark 2.3. This is non-connective K-theory. One can also construct connective K-theory via a similar procedure – take the universal additive invariant and follow the same procedure as above.

This construction allows us, for example, to define K-theory of a ring since $\mathrm{Perf}(R) \in \mathrm{Cat}_\infty^{ex}$. This construction recovers the classically defined K-groups we already constructed, and gives us $K_n(R)$ for $n > 2$.

Remark 2.4. In particular, we have now constructed $K(R)$ for an $\mathbb{E}_\infty$-ring R (including discrete rings) *as a ring spectrum*. Where does the ring structure come from?

The universal localising invariant is strongly symmetric monoidal. Maps out of the unit is always a lax symmetric monoidal functor. So the composition K is lax symmetric monoidal. For R an $\mathbb{E}_\infty$ ring, we know $\mathrm{Perf}(R) \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{ex})$ and so $K(R) \in CAlg(\mathrm{Sp})$.

Note that Morita invariance is also built into this definition of K-theory.

How do we build the universal additive invariant? It's actually quite simple, compose the following functors:

- idempotent completion $\mathrm{Cat}_\infty^{ex} \to \mathrm{Cat}_\infty^{perf}$
- Yoneda embedding y to presheaves of spectra
- restrict to the subcategory of compact objects
- localise at maps of the form $y(B)/y(A) \to y(C)$ for split exact sequences
- stabilise.

This gives a functor $U_{loc} : \mathrm{Cat}_\infty^{ex} \to \mathrm{NMot}_{loc}$ which is universal in the sense that for any stable presentable ∞-category $\mathscr{D}$, there is an equivalence of ∞-categories

$$\mathrm{Fun}^L(\mathrm{NMot}_{add}, \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}_{add}(\mathrm{Cat}_\infty^{ex}, \mathscr{D}).$$

For details and the proof, see [3], Section 6. The universal localizing invariant can be further constructed from U_{add}, details in [3].

What is useful about the universal perspective on K-theory? One reason is that this construction makes it clear that K-theory is symmetric monoidal and universal additive (or localizing for non-connective). Another reason is that we now have a representability result for K-theory, which gives us a perhaps more tractable perspective for studying maps to other additive invariants. For example, recent progress has been made studying the trace map $K \to THH$.

Warning: Although constructing the universal additive/localizing invariant is easy, proving that we can recover K-theory from it is more difficult. The presentation I have given is ahistorical – of course, all the other definitions of K-theory predate this formal and universal one.

2.2. **Three more constructions.** There are two constructions of K-theory I won't discuss in detail for now. The Q-construction (Q for Quillen) is for exact categories, and the $+$-construction is for (discrete) rings. These two constructions agree for a ring R (applying the Q construction to its category of projective modules). We may discuss the $+$ construction in some detail later as this is really the only way one can do computations (e.g. the K-theory of a finite field).

Let us briefly discuss K-theory for symmetric monoidal ∞-categories, which looks like the group completion we did in this setting to get K_0. We take the composition

$$K : \mathrm{CMon}(\mathrm{Cat}_\infty) \to \mathrm{CMon}(\mathrm{Gpd}_\infty) \to \mathrm{AbGrp}(\mathrm{Gpd}_\infty) \hookrightarrow \mathrm{Sp}$$

where the first functor is the core and the second functor is ∞-group completion, which is left adjoint to the forgetful functor from abelian groups to commutative monoids. But then abelian group-like objects in ∞-groupoids are just connective spectra. This produces connective K-theory for symmetric monoidal ∞-categories.

If R is a discrete ring, then applying this construction to its category of finitely generated projective modules reproduces connective K-theory of the ring as we have previously defined it.

Remark 2.5. Although group completing in this way is aesthetically appealing, it is actually subtly wrong in many contexts. The problem is that we want K-theory to formally split all exact sequences, even the non-split ones. This construction will produce the right thing for a ring R because all short exact sequences of finitely generated projective modules are split already. But it won't behave correctly if we try to do something similar for scheme – there's no analogue for projective modules over a ring which could possibly work here.

This definition gives us that $K(\text{FinSet}) = \mathbb{S}$. This is basically because the $\mathbb{E}_\infty$ space (space = ∞-groupoid) obtained for FinSet is the free such space on one generator, which group-completes to the free group-like $\mathbb{E}_\infty$-algebra on one generator, i.e. the sphere spectrum.

2.3. **Waldhausen S-construction.** Let's talk about Waldhausen's S-construction. This is a concrete way to build the connective K-theory spectrum of a Waldhausen category. Throughout this section, let $\mathcal{C}$ be a Waldhausen category.

Definition 2.6. Let $S_n\mathcal{C}$ be the category whose objects are commutative diagrams of the form

$$\begin{array}{ccccccccc}
 & & & & & & & & A_{n-1n} \\
 & & & & & & & & \twoheaduparrow \\
 & & & & A_{23} & \hookrightarrow & \dots & \hookrightarrow & A_{2n} \\
 & & & & \twoheaduparrow & & & & \uparrow \\
 & & A_{12} & \hookrightarrow & A_{13} & \hookrightarrow & \dots & \hookrightarrow & A_{1n} \\
 & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
A_1 & \hookrightarrow & A_2 & \hookrightarrow & A_3 & \hookrightarrow & \dots & \hookrightarrow & A_n
\end{array}$$

in $\mathcal{C}$, and whose morphisms are morphisms at each vertex of the diagram, which commute with everything. In the above diagram, inclusions are used to denote cofibrations in $\mathcal{C}$, and surjections the corresponding cofibre. The A_{ij} come from the cofibration sequence

$$A_i \hookrightarrow A_j \twoheadrightarrow A_{ij}$$

for $1 \leq i < j \leq n$. (An object is really given by the sequence $A_1 \hookrightarrow \dots \hookrightarrow A_n$ and everything else in the diagram is formed by taking cofibres compatibly. We might think of an object as the sequence $0 \hookrightarrow A_1 \hookrightarrow \dots \hookrightarrow A_n$.)

Now, each $S_n\mathcal{C}$ is itself a Waldhausen category, and they assemble into a simplicial Waldhausen category. The weak equivalences in $S_n\mathcal{C}$

are simply pointwise weak equivalences, and it is enough to check this on the non-quotient terms A_i. The cofibrations are morphisms which become cofibrations after mapping to $S_2\mathcal{C}$ in any of the possible ways to do so.

To assemble this into a simplicial category, we need to provide face and degeneracy maps. The degeneracy map s_i is given by inserting an extra copy of A_i. With the rows of the diagram of an object of $S_n\mathcal{C}$ numbered from 0 to $n-1$, the face map f_n is given by deleting row i and the column containing A_i (i.e. 'composing through' A_i) and reindexing the diagram as necessary.

Example 2.7. $S_0\mathcal{C} = *$ because the only object here is a sequence of length 0 in $\mathcal{C}$, and there are no nontrivial morphisms. $S_1\mathcal{C} = \mathcal{C}$ where we think of an object A of $\mathcal{C}$ as the cofibration $0 \hookrightarrow A$. $S_2\mathcal{C}$ has objects of the form $A \hookrightarrow B \twoheadrightarrow B/A$ coming from cofibrations in $\mathcal{C}$.

Now, consider the bisimplicial set $N(wS_\bullet\mathcal{C})_\bullet$, where $wS_\bullet\mathcal{C}$ is the subsimplicial category of $S_\bullet\mathcal{C}$ formed by taking only those morphisms in each $S_n\mathcal{C}$ which are weak equivalences. We can form its geometric realisation as the diagonal simplicial set $N(wS_n\mathcal{C})_n$.

Definition 2.8. $K(\mathcal{C}) = \Omega^\infty N(wS_\bullet\mathcal{C})_\bullet$, for $\mathcal{C}$ Waldhausen.

Restricting to weak equivalences here is going to give us the weak equivalence relations we want in K-theory, and the simplicial construction will split cofibration sequences. As a sanity check, we'll make sure that this recovers the definition of K_0 we had previously.

Lemma 2.9. $\pi_1 N(wS_\bullet\mathcal{C})_\bullet = K_0(\mathcal{C})$.

This is why we are taking loops to shift everything back down into the right degree. ($S_0\mathcal{C} = *$ so we were always going to need a shift.)

Proof. Let $G_\bullet$ be the diagonal simplicial set $G_n = N(wS_n\mathcal{C})_n$. We have $G_0 = *$, so every path in G_1 is a loop. Hence we can compute $\pi_1 G_\bullet$ with elements of G_1 as generators and elements of G_2 as relations, where the relation for a 2-simplex A is $d_0(A) + d_2(A) \simeq d_1(A)$. Now $G_1 = \{A \xrightarrow{\sim} B\}$. Also G_2 has elements formed from length 2 compositions of weak equivalences in $S_2\mathcal{C}$, i.e. they are diagrams

$$\begin{array}{ccccc} A' & \hookrightarrow & A & \twoheadrightarrow & A'' \\ \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\ B' & \hookrightarrow & B & \twoheadrightarrow & B'' \\ \downarrow{\sim} & & \downarrow{\sim} & & \downarrow{\sim} \\ C' & \hookrightarrow & C & \twoheadrightarrow & C'' \end{array}$$

in $\mathcal{C}$. What relation does each such diagram give us? Well, in $S_2\mathcal{C}$ we have face maps for $X = A' \hookrightarrow A \twoheadrightarrow A''$

$$d_0(X) = 0 \hookrightarrow A'', d_1(X) = 0 \hookrightarrow A, d_2(X) = 0 \hookrightarrow A'.$$

Combining these with the face maps from the nerve, we get that the relation coming from an element of $S_2\mathcal{C}$ is

$$[A \xrightarrow{\sim} C] = [A' \xrightarrow{\sim} B'] + [B'' \xrightarrow{\sim} C''].$$

How do we use this to get what we want? First, make all 3 rows $0 \hookrightarrow A \twoheadrightarrow A$. The relation implies $[0 \xrightarrow{\sim} 0] = 0$. Now, given a weak equivalence $A \xrightarrow{\sim} B$ make the top row $0 \hookrightarrow A \twoheadrightarrow A$ and the bottom two rows the same for B. Gives the relation $[A \xrightarrow{\sim} B] = [B \xrightarrow{id} B]$. Something similar with 0 in the rightmost column gives $[A \xrightarrow{\sim} B] = [A \xrightarrow{id} A]$.

Now we know that $\pi_1 G$ is generated by $[A \xrightarrow{id} A]$ up to weak equivalence of the object A. We get the desired cofibration relation from taking a diagram with all three rows the same, some given cofibration sequence. □

The Waldhausen construction can be generalised to Waldhausen ∞-categories, see [4] for details.

2.4. **Non-connective K-theory spectrum.** To finish off this section, we would like to extend the S-construction to a non-connective spectrum which captures the negative K-groups. In fact, the approach we outline will allow us to construct a non-connective K-theory spectrum starting from any of the functorial constructions of the connective K-theory spectrum.

Suppose we have K a functor from rings to connective spectra. We are going to build a diagram of spectra where π_0 recovers the diagram of groups we used to classically construct negative K-theory. The limit of this diagram will give us non-connective K-theory.

As with K_0, we have a splitting $K(R[x]) \cong K(R) \oplus NK(R)$. So we have a homotopy pushout

$$\begin{array}{ccc} K(R) & \longrightarrow & K(R[x]) \\ \downarrow & & \downarrow \\ K(R[x^{-1}]) & \longrightarrow & K(R) \oplus NK(R) \oplus NK(R) \end{array}$$

and the pushout maps to $K(R[x, x^{-1}])$ because the other three terms of the diagram do. Let $LK(R)$ be the cofibre of the map $K(R[x]) \sqcup_{K(R)} K(R[x^{-1}]) \to K(R[x, x^{-1}])$. Then we get a natural cofibration sequence (natural in K and R)

$$\Omega LK(R) \to K(R[x]) \sqcup_{K(R)} K(R[x^{-1}]) \to K(R[x, x^{-1}]) \to LK(R)$$

which, after applying π_n, recovers the exact sequence

$$\begin{aligned} \pi_{n+1} LK(R) &\to K_n(R) \oplus NK_n(R) \oplus NK_n(R) \\ &\to K_n(R) \oplus NK_n(R) \oplus NK_n(R) \oplus K_{n-1}(R) \to \pi_n LK(R) \end{aligned}$$

and so after matching up terms, we see that $K_{n-1}(R) \cong \pi_n LK(R)$ for $n \geq 0$. We also have $\pi_n LK(R) = 0$ for $n < 0$. So after desuspending to make the homotopy groups match up, $\Omega LK(R)$ is (-1)-connective. We get a map $K(R) \to \Omega LK(R)$ which induces an isomorphism on π_n for $n \geq 0$.

Now we just repeat this inductively to get something $(-n)$-connective for each n, and taking the limit gives non-connective K-theory.

3. K-theory of finite fields

3.1. **Quillen's + construction.** We first recall the + construction, and then use it to compute K-theory of finite fields. The actual computation will be a sketch with many details omitted. This construction is by analogy to the definition of $K_1(R)$ for a ring as $GL(R)/E(R)$.

Let G be a topological group. Recall that its classifying space BG has fundamental group G, and all higher homotopy groups vanish. Also BG is connected.

Definition 3.1. Let $BGL(R)^+$ be a CW complex equipped with a map $BGL(R) \to BGL(R)^+$, satisfying

(1) the induced map $GL(R) = \pi_1 BGL(R) \to \pi_1 BGL(R)^+$ is surjective with kernel $E(R)$;
(2) $H_*(BGL(R); M) \xrightarrow{\cong} H_*(BGL(R)^+; M)$ for any $K_1(R)$-module M.

Clearly (1) ensures that $\pi_1 BGL(R)^+ = K_1(R)$. For $n \geq 1$, define $K_n(R) = \pi_n BGL(R)^+$.

We can build a model for $BGL(R)^+$ as a CW complex; the definition already suggests the correct procedure. There's also a universal construction: $BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces.

You may notice that we can always choose $BGL(R)^+$ to be connected, in which case $\pi_0 BGL(R)^+ = *$ may not agree with $K_0(R)$. To remedy this, we take $K(R) = K_0(R) \times BGL(R)^+$ to be the disjoint union of $K_0(R)$ copies of $BGL(R)^+$. This clearly preserves the higher homotopy groups and fixes K_0. This construction is really as a topological space, and doesn't give us the K-theory spectrum. (But it has all the right homotopy groups.)

Now we sketch how Quillen computed the K-theory of finite fields.

Theorem 3.2 (Quillen)**.** *There is a fibre sequence*

$$BGL(\mathbb{F}_q)^+ \longrightarrow BU \xrightarrow{1-\psi^q} BU.$$

Where does the map $BGL(\mathbb{F}_q)^+ \to BU$ come from? Well, we have a map $(I_n - \mathrm{id}) : \mathrm{GL}_n(\mathbb{F}_q) \to U$ coming from (the different between) the trivial and the standard n-dimensional complex representations of $\mathrm{GL}_n(\mathbb{F}_q)$. This induces a map on classifying spaces $B\,\mathrm{GL}_n(\mathbb{F}_q) \to BU$ which is compatible (up to homotopy) with the inclusion $BGL_n(\mathbb{F}_q) \hookrightarrow$

$BGL_{n+1}(\mathbb{F}_q)$, so altogether we get a map $BGL(\mathbb{F}_q) \to BU$. This induces a map on $BGL(\mathbb{F}_q)^+$ by the universal property of $+$, because BU is an H-space.

From the homotopy fibration in the theorem, to compute $K_n(\mathbb{F}_q) = \pi_n(BGL(\mathbb{F}_q)^+)$ we need only determine what $\psi^q : BU \to BU$ does on homotopy groups. Homotopy of BU is essentially controlled by Bott periodicity, and ψ^q turns out to be multiplication by q^i on $\pi_{2i}(BU)$. We deduce:

Corollary 3.3. *The K-groups of a finite field $\mathbb{F}_q$ are*

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^{\frac{n+1}{2}} - 1) & n \ odd, n \geq 1 \\ \mathbb{Z} & n = 0 \\ 0 & n \ even, n \geq 2. \end{cases}$$

Note that only the groups for $n \geq 1$ come from Quillen's Theorem, i.e. from $\pi_n BGL(\mathbb{F}_q)^+$. We already computed K_0 and K_1, and what we computed for K_1 agrees with the fundamental group of $BGL(\mathbb{F}_q)^+$. Of course, the negative K-groups vanish for any regular ring and in particular for any field.

References

[1] C. A. Weibel, *The K-book: An Introduction to Algebraic K-theory*, vol. 145. American Mathematical Soc., 2013.

[2] T. Brazelton, "K-theory of infinity categories," *talk notes*, 2020.

[3] A. J. Blumberg, D. Gepner, and G. Tabuada, "A universal characterization of higher algebraic K-theory," *Geometry & Topology*, vol. 17, no. 2, pp. 733–838, 2013.

[4] C. Barwick, "On the algebraic K-theory of higher categories," *Journal of Topology*, vol. 9, no. 1, pp. 245–347, 2016.