RATIONAL TANGLE REPLACEMENT AND ITS CONSEQUENCES

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Abstract. We analyse how rational tangle replacement in a link changes the double branched cover of S^3, branched over the link. We use the results of this analysis to show that a 3-manifold M is a 2-fold cyclic branched cover of S^3, branched over a link, if and only if it is the result of surgery on a strongly-invertible link in S^3. We give an explicit description of the surgery coefficients corresponding to some rational tangle replacement, and conclude by discussing the implications of these results in relation to the Poincaré Conjecture.

1. Introduction

The theory I will be discussing in this paper was developed prior to Perelman's proof of the Poincaré Conjecture, and is motivated[1] in large part with the goal of restricting the class of possible counterexamples.

Our discussion will instead be motivated by the following related[2] conjectures.

Conjecture 1.1. Every closed, orientable, simply-connected 3-manifold is a 2-fold cyclic branched cover of S^3.

Conjecture 1.2. Every closed, orientable 3-manifold is a 2-fold branched cyclic cover of a 2-fold branched cyclic cover of... of a 2-fold branched cyclic cover of S^3.

Since every (closed, orientable, connected) 3-manifold can be obtained by surgery on a link in S^3, we study these conjectures by determining what surgeries give 2-fold cyclic branched covering spaces of S^3. In fact, we give a method for converting any 2-fold cyclic branched covering space, branched over a link, into surgery over another link with coefficients computed explicitly.

To obtain such a conversion, we use rational tangle replacement, whereby part of a link which is a rational tangle is modified to a different rational tangle. This changes the double cover branched over the link by a surgery, and since any link may be transformed into the unknot by several such replacements we can convert the double branched cover over any link into surgery.

Rather than give a discussion of the theory of rational tangles here, we direct the reader to Section 2 of [2]. In particular, before reading further we suggest the reader familiarise themself with the process by which an associated rational number may be obtained from any rational tangle, by choosing an equivalent basic rational tangle. Throughout this paper, we view all rational tangles as being (properly) embedded in a ball, as two boundary-parallel arcs.

2. Rational Tangle Replacement

We are interested in studying which 3-manifolds occur as 2-fold cyclic covers of S^3, branched over some link. In light of the result[3] that every closed, orientable, connected 3-manifold can be obtained via Dehn surgery on some link in S^3, we would like to determine which surgeries correspond to 2-fold cyclic covers of S^3. In this section, we are interested in taking a 2-fold cyclic cover branched over some link and analysing how the 2-fold cover changes when we modify the link locally. It turns out that modifying the link by a rational tangle has the effect of changing the 2-fold cover by surgery on a knot, with the surgery coefficient given by the rational number corresponding to the rational tangle we replaced.

[1] In particular, the paper [1] by Montesinos which this is based on is chiefly concerned with trying to reduce the search for a counterexample to Poincaré to 2-fold cyclic covers of S^3.

[2] If either version of this conjecture is true, then it can be shown that if a counterexample to Poincaré exists, there must also be a counterexample that is a 2-fold cyclic cover of S^3. Hence proving either of these would give the desired reduction. Note also that Poincaré implies Conjecture 1.1 since the only such space is S^3 itself.

[3] Due to Lickorish and Wallace, see [3].

Definition 2.1. Let L be a link in S^3. We denote by $\tilde{L}$ the 2-fold cyclic cover of S^3, branched over L.

We next make precise the idea of modifying a link by a rational tangle.

Definition 2.2. Let L be a link in S^3 and T' be a rational tangle. Choose some $B^3 \subset S^3$ which intersects L in a rational tangle T, meaning that $L \cap B^3$ contains exactly two arcs and these two arcs form a rational tangle inside the ball. Then we can modify the link L by removing the ball containing T and gluing T' back in, so that the free ends match up. This is called a *rational modification of L (by T')*.

Rational modifications can be used to transform any link into a knot, since choosing T, T' to be the trivial vertical and horizontal tangles respectively, where the arcs of T belong to distinct components of the link, we may reduce the number of components by one. Rational modifications may also be used to effect crossing changes, where T, T' are the basic horizontal tangles $+1$ and -1. In this way any link may be transformed into the unknot by finitely many rational modifications. See Figure 1 for an illustration.

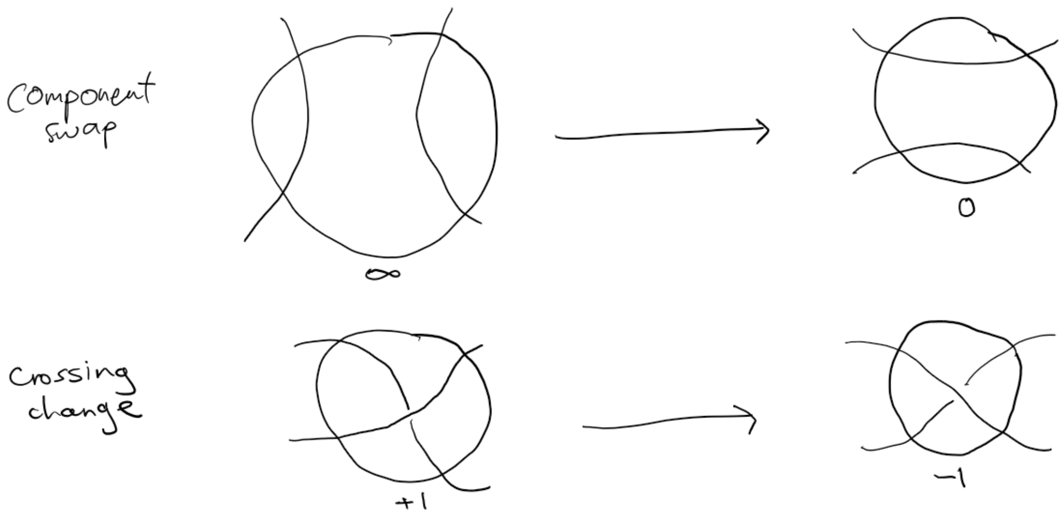

FIGURE 1. Two rational modifications, by a combination of which any link may be transformed into the unknot

We now have the language to state the main result for this section, usually called the Montesinos Trick.

Theorem 2.3 (inspired by [4])**.** *Let L be a link in S^3, and L' be another link obtained from L via rational modification. Then $\tilde{L}'$ is obtained from $\tilde{L}$ by Dehn surgery on a knot.*

Moreover, if L'' is obtained from L by a collection of n rational modifications occuring in disjoint neighbourhoods, then $\tilde{L}''$ is obtained from L by surgery on a link with n components.

Concretely, if $B^3 \subset S^3$ is the neighbourhood in which the rational modification from L to L' occurs, then we obtain $\tilde{L}'$ from $\tilde{L}$ via surgery on the solid torus in $\tilde{L}$ which is the lift of this B^3 to the 2-fold cover.

Proof. Because we build rational tangles by twisting the free ends around one another, there is a homeomorphism from $B^3 \setminus T'$ to $B^3 \setminus T$. Hence the double cover of the ball branched over T is homeomorphic to the double cover of the ball branched over T', and both are the same as the double cover of the ball branched over the trivial (vertical or horizontal) tangle. This is the solid torus.

Then $\tilde{L}'$ is related to $\tilde{L}$ by cutting out the solid torus which covers the ball containing T and gluing back in the solid torus which covers the ball containing T'. The solid torus we cut out is a neighbourhood of a knot in $\tilde{L}$, so we obtain $\tilde{L}'$ from $\tilde{L}$ by performing surgery along this knot. □

One can in fact compute the surgery coefficients explicitly. For simplicity, suppose L is the unknot and L' is a rational tangle T with the top free ends joined together and the bottom free ends joined together (that is, obtained from L by replacing the trivial vertical tangle by T). Then $\tilde{L} = S^3$ and the ball in which we performed the replacement lifts to an unknotted solid torus, so $\tilde{L}'$ is given by surgery on the unknot. To find the coefficient of this surgery, we find a meridional disc in the solid torus corresponding to T.

Observe that the trivial (horizontal or vertical) tangle has a properly-embedded disc which is disjoint from and which separates the two arcs. We can for example pick this disc to divide the ball into two hemispheres (see Figure 2). In fact, every rational tangle has a separating disc like this. To obtain such a disc, we simply start with the separating disc for the trivial tangle and build the desired rational tangle by succesively twisting adjacent free ends around one another, and deforming the disc at the same time. Each such twist is a homeomorphism of the ball, so it carries a separating disc to a separating disc.

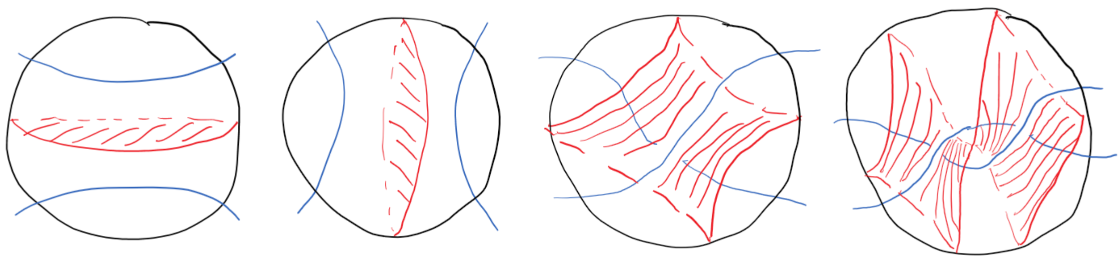

FIGURE 2. Separating discs for the trivial horizontal and vertical tangles respectively, and for the more interesting tangles +1 and +2

The separating disc is disjoint from the two arcs, so lifts to two disjoint properly-embedded discs in the double cover. Cutting the ball along the separating disc produces two balls, each with a single properly-embedded arc. Cutting along both of the lifted discs will therefore cut the solid torus into two connected components, each of which is the double cover of a ball with a single embedded arc, that is a ball. Hence each lift of the boundary of a separating disc is a meridian in the solid torus, and in fact the two lifts are homologous. Therefore either lift of the boundary of a separating disc will give us the surgery coefficient.

Consider the trivial vertical tangle with its separating disc, and decorate the surface of the ball with the boundary of the separating disc for T. Let this boundary curve be γ_T. Lifting this to the double cover branched over the trivial vertical tangle, the trivial separating disc lifts to the standard meridian while γ_T lifts to a curve which is meridional in the double cover branched over T. We know that ∂B^3 lifts to the T^2 surgery surface in the double cover, since the lift is just S^2 branched over four points. Then the surgery coefficient is given by lifting γ_T to the double cover of B^3 branched over the trivial tangle, and computing the slope of the resulting curve in T^2. We have the following general result.

Theorem 2.4 (inspired by [4])**.** *Rational modification of the unknot in S^3, changing the trivial vertical tangle into a rational tangle with associated rational number s, corresponds to s-surgery on the unknot in the double branched cover.*

Proof sketch. We prove this for basic tangles, by induction. Suppose we have the separating disc corresponding to a basic tangle T with associated rational number $s = \frac{p}{q}$, and we know that the boundary of this disc lifts to the curve $p\mu + q\lambda$ in the solid torus which is the double cover.

Since we started with the trivial vertical tangle, one half of the double cover is formed by slicing open the ball along two discs, bounded between the two vertical strands and the surface of the ball, and then gluing two cylinders obtained in this way together at their ends. So twisting the separating disc for T horizontally on the left or right has the effect of twisting each half of the torus by a half-twist in the meridional direction. Hence if we modify T by adding an integral horizontal twist by n, this has the effect of twisting the boundary of the disc on one half of the solid torus by n half-twists, in the meridional direction.

Thus each strand of the curve $p\mu + q\lambda$ is twisted by n half twists on each half of the torus, so each strand gains n extra meridians in total. We have q copies of the longitude to start with, and so there are q strands which each gain n full twists. Therefore the resulting curve is $(p+nq)\mu+q\lambda$ which gives the surgery coefficient $s+n$. This is the rational number corresponding to T after adding n horizontal twists.

A symmetric argument with the roles of the meridian and longitude switched shows that adding a vertical twist by n to T has the effect of changing the meridional curve from $p\mu+q\lambda$ to $p\mu+(np+q)\lambda$. This gives a surgery coefficient of

$$\frac{p}{np+q} = \frac{1}{n+\frac{q}{p}} = \frac{1}{n+s^{-1}},$$

which is the rational number corresponding to T after adding n vertical twists. □

We have now done most of the work we need to take a 2-fold cyclic cover of S^3 branched over a link, and construct the same space using surgery in S^3 on some other link. Before we discuss the main theorem, we illustrate the computational utility of our surgery coefficient result.

Example 2.5. We know by the previous theorem that starting with the unknot in S^3, if we replace the trivial vertical tangle by a vertical twist by n then this corresponds to $1/n$-surgery in the double cover, that is $1/n$-surgery on the unknot in S^3.

But replacing the trivial vertical tangle by a vertical twist gives us back the unknot (see Figure 3) and so the double cover is still S^3. Hence $1/n$ surgery on the unknot in S^3 gives S^3, for any integer n. This can be shown more explicitly in other ways, but the relationship between double covers branched over a link and surgery on a link makes this result immediate.

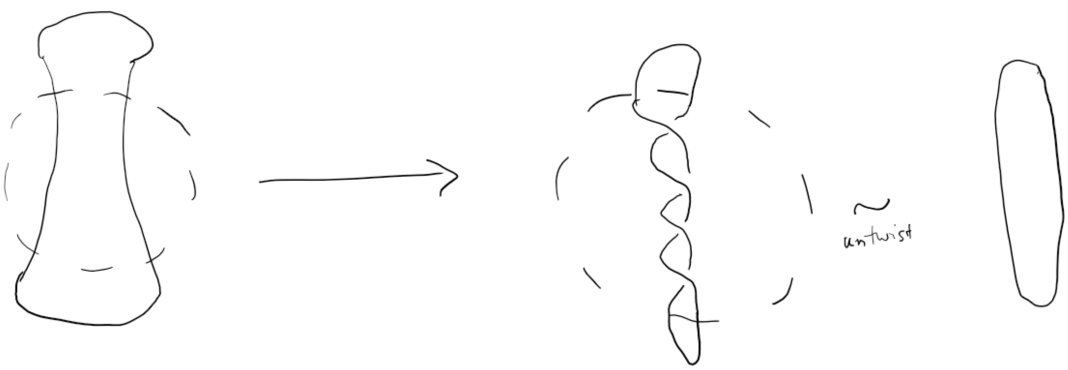

Figure 3. A trivial rational replacement giving a nontrivial surgery coefficient

3. Converting Between Double Branched Covers of S^3 and Surgery on a Link

In this section, we classify what which links in S^3 have surgeries giving rise to 2-fold cyclic branched covers of S^3. It turns out that this is a property of the link, not the particular surgery coefficients.

Definition 3.1. A link L in S^3 is called *strongly-invertible* if there is some axis E of S^3 which intersects each component of L exactly twice and such that a rotation of 180° around this axis restricts to an automorphism in each component of L (or if some link ambient-isotopic to L has this property).

In particular, a strongly-invertible link comes with an orientation-preserving involution r of S^3, namely the rotation, which restricts in each component of L to an involution with exactly two fixed points. These properties are usually taken as the definition of a strongly-invertible link, but it turns out that by modifying such an involution by an isotopy of S^3, one may always obtain a rotation. Hence the two definitions are equivalent, and we will prefer working with concrete rotations.

Theorem 3.2 ([1], Thm 1). *Let M be a closed, orientable 3-manifold. Then M is a 2-fold cyclic branched cover of S^3, branched over a link, if and only if M can be obtained by surgery on a strongly-invertible link in S^3.*

Moreover, if M is obtained by surgery on a strongly-invertible link of n components, then M is a 2-fold cyclic cover branched over a link of at most $n+1$ components.

Proof. One implication follows from rational tangle replacement. Let $M = \tilde{L}$ for some link L. We may perform finitely many rational modifications in disjoint balls to change the unknot into L. The double cover of S^3 branched over the unknot is S^3, so by Thm 2.3 we may perform surgery along finitely many disjoint solid tori in S^3 to obtain M. Let these solid tori be $K_1, \ldots, K_n$.

The 2-fold (branched) covering map $M \to S^3$ can be interpreted as an involution of M by sending each point in M to the other point in the same fibre. This restricts to an involution on the complement of the K_i, from which we obtain an involution on S^3, thought of as M before the surgery, by extending across the tori taken out.

The set of fixed points of this involution lying outside these solid tori is the lift of the branch locus L, which intersects each ∂K_i in four points. The set of fixed points intersects each solid torus in two properly-embedded arcs. Then choosing an appropriate core of each K_i (intersecting the set of fixed points twice) we obtain a knot which is strongly-invertible w.r.t. this involution. Taking these cores as components, we obtain the desired strongly-invertible link.

We will prove the converse for the case $n = 1$, that is when L is a knot. The general case is much the same.

Let M be obtained by surgery on a strongly-invertible knot K in S^3. Fix an axis of rotation E and rotation map r. Let $U(K)$ be a neighbourhood of K on which the rotation restricts to an involution, and we will perform surgery by cutting out $U(K)$. Let V be the solid torus we glue back in, and pick an axis E' passing through V so that V has a rotational symmetry r' about E'.

Let ϕ be the surgery gluing map, from ∂V to $\partial U(K)$. By modifying ϕ by an isotopy, we may assume that ϕ is compatible with the symmetries r and r', so that

$$\phi(r'|_{\partial V}) = (r|_{\partial U(K)})\phi.$$

Then the space M obtained by surgery has a symmetry r'' which is given by r' on V and r on $S^3 \setminus U(K)$, since these agree on the boundary. This makes M a 2-fold cyclic branched cover of its orbit space under this symmetry.

The orbit space of V under r'' is a B^3, with branching set two properly embedded arcs in V, given by $E' \cap V$. The orbit space of $U(K)$ under r is a B^3 (or a solid cylinder), since K intersects E in exactly two points, giving the two end-caps of the cylinder in the orbit space. Therefore the orbit space of $S^3 \setminus U(N)$ under r is S^3 minus a ball, and the branching set is E with two segments removed, where the two segements are those forming $E \cap U(N)$. There is only one way to glue a ball to $S^3 \setminus B^3$, so the orbit space of M is S^3.

Finally, the branching set of M is given by $E \setminus (E \cap U(N)) \cup (E' \cap V)$ which is a loop with two segments cut out and two segments glued in. This is a link with at most two components. □

This theorem has a couple of nice corollaries. Suppose we have a family of strongly-invertible links in S^3 which are known to have property P — that is, nontrivial surgery on such links cannot produce a simply-connected space. Then Theorem 3.2 gives a corresponding family of links whose double branched covers cannot be simply-connected. Given that the Poincaré Conjecture has now been established, this tells us that the double branched cover of S^3 branched over any link in S^3 that isn't the unknot cannot be simply-connected, since surgery along any strongly-invertible link must give either S^3 or a space which is not simply-connected, but the unknot is the only link whose double branched cover is S^3.

Corollary 3.3. *Every non-trivial link L satisfies $\pi_1(\tilde{L}) \neq 0$, if and only if every strongly-invertible link has property P.*

As a second application, recall our original motivating Conjecture 1.1. In light of the theorem, any counterexample must be obtained by surgery on a link which is not strongly-invertible, and similarly for Conjecture 1.2.

To conclude, we discuss a generalisation of Theorem 3.2 which can be used to establish Conjecture 1.2 for a large class of knots.

Let L be a link in S^3 with an orientation-preserving involution r of S^3 which restricts to an involution on each component of L. So we want L to be strongly-invertible, except we drop the requirement that r has exactly two fixed points in each component of L. Let L' be the link formed from those components of L on which r has some number of fixed points other than 2. Then the involution r defines a 2-fold cyclic branched cover $p : S^3 \to S^3$, and we have the following.

Theorem 3.4 ([1], Thm 2)**.** *Every manifold obtained by surgery on a link L is a 2-fold cyclic branched cover of a manifold obtained by surgery on* $p(L')$.

We refrain from indicating the proof. It uses many of the same ideas as the proof of Theorem 3.2, which is the special case when L' is empty. We finish with an application.

Example 3.5. Consider the knot $K(3,5,7)$ depicted in Figure 4, with involution the 180° rotation around the axis shown. The knot has no fixed points under this rotation, and L' in this case consists of its single component. We see that $p(K(3,5,7))$ is the unknot, so by Theorem 3.4 any manifold obtained by surgery on $K(3,5,7)$ is a 2-fold cyclic branched cover of a manifold obtained by surgery on the unknot. But the unknot is strongly-invertible, so by Theorem 3.2 surgery on the unknot gives a 2-fold cyclic branched cover of S^3.

Hence any manifold obtained by surgery on $K(3,5,7)$ is a 2-fold cyclic branched cover of a 2-fold cyclic branched cover of S^3.

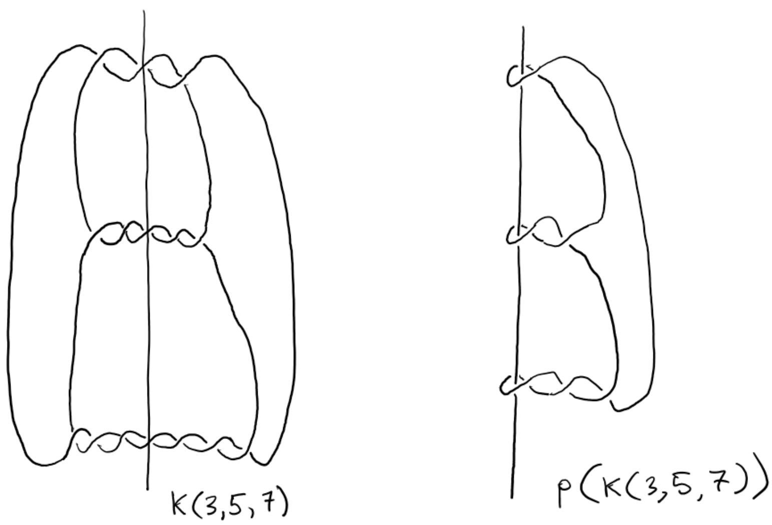

Figure 4. A knot with no fixed points under the rotation about the axis shown, and its projection

4. References

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