Introduction to Quiver Representations

Isabel Longbottom

Australian National University

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Representations of quivers

Definition

A quiver Q is a finite directed graph, with vertices Q_0 and edges Q_1. A *representation* v of Q is a choice of finite-dimensional vector space $v(q)$ for each $q \in Q_0$, and linear map $v(\alpha)$ for each $\alpha \in Q_1$.

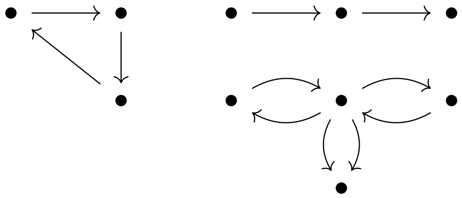


Figure 1: Three quivers

A representation is *simple* if it has no proper nonzero sub-representations, and *indecomposable* if it can't be written as the direct sum of two nonzero representations.

Example: The A_2 quiver

Consider the following quiver:

$$\bullet \longrightarrow \bullet$$

What are the indecomposables?

$$k \to 0, \qquad 0 \to k \qquad \text{and} \qquad k \xrightarrow{\sim} k$$

Any representation can be decomposed in the following manner:

$$V_0 \xrightarrow{\alpha} V_1 \cong (\ker \alpha \to 0) \oplus (0 \to W_1) \oplus \left(W_0 \xrightarrow{\sim} \operatorname{im} \alpha \right)$$

where $V_0 = (\ker \alpha) \oplus W_0$ and $V_1 = (\operatorname{im} \alpha) \oplus W_1$.

Warm-up: What are the simple representations?

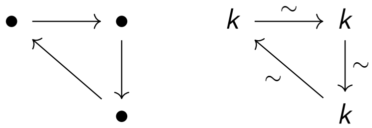
Theorem

If Q is an acyclic quiver (has no oriented cycles), then any simple representation v satisfies $v(q) = 0$ for every $q \in Q_0 \setminus \{q_i\}$, and $v(q_i) = k$ for a single vertex q_i.

The acyclicity hypothesis is necessary.

Example

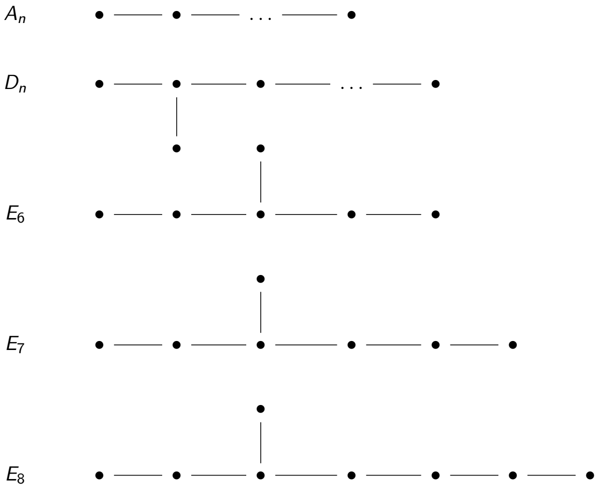
A cyclic quiver and a simple representation thereof.



Dynkin quivers

Definition

A Dynkin quiver is one whose underlying graph is of the following form:



Indecomposable representations of finite type quivers

Theorem (Gabriel)

The quivers with finitely many (isomorphism classes of) indecomposable representations are precisely the Dynkin quivers. Moreover, the indecomposable representations of such a quiver are in bijection with the positive roots in the root system corresponding to the underlying Dynkin diagram, with bijection given by taking the dimension vector.

Example ($A_3, \bullet \to \bullet \to \bullet$)

6 indecomposables, 3 of which are simple:

$$k \to 0 \to 0, \qquad 0 \to k \to 0, \qquad 0 \to 0 \to k,$$
$$k \overset{\sim}{\to} k \to 0, \qquad 0 \to k \overset{\sim}{\to} k, \qquad k \overset{\sim}{\to} k \overset{\sim}{\to} k.$$