Reflections of Quiver Representations

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Representations of quivers

Definition

A quiver Q is a finite directed graph, with vertices Q_0 and edges Q_1. A *representation* V of Q is a choice of finite-dimensional vector space $V(q)$ for each $q \in Q_0$, and linear map $V(\alpha)$ for each $\alpha \in Q_1$.

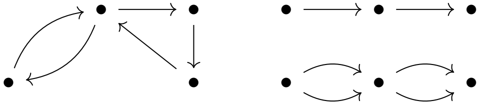


Figure 1: Three connected quivers

Q-representations form an abelian category, Rep Q.
A representation is *simple* if it has no proper nonzero sub-representations, and *indecomposable* if it can't be written as the direct sum of two nonzero representations.
The *underlying graph* $\Gamma(Q)$ is obtained by forgetting the orientations of arrows.

Some orientation-independent results

The following aspects of the representation theory of Q depend only on $\Gamma(Q)$:

- The simple representations of Q, if $\Gamma(Q)$ is acyclic.
- (Gabriel's theorem) Whether Q has finitely many isomorphism classes of indecomposable representations.
- (Kac's theorem) The possible dimension vectors of indecomposable representations.

Recovering Q from simples

Theorem

Let Q, Q' be acyclic quivers. Then $\operatorname{Rep} Q \simeq \operatorname{Rep} Q'$ if and only if $Q \cong Q'$ as directed graphs.

Q can be recovered from the homological algebra of simple objects.

- Simple representations are in bijection with vertices of Q; the simple at $x \in Q_0$ is denoted S_x.
- $\dim \operatorname{Ext}(S_x, S_y)$ is the number of arrows from x to y in Q.

This works even when Q contains cycles, by looking at the 1-dimensional simple objects.

Derived category of quiver representations

Given an acyclic quiver Q, we consider the category $\mathcal{D}^b(\operatorname{Rep} Q)$. Objects are bounded chain complexes of Q-representations, up to cohomology.
Think of objects as $V = (V_n)_{n\in\mathbb{Z}}$ where each V_n is a Q-representation, and all but finitely many components are zero. Morphisms $V \to W$ have components of two forms:

- $\phi_n \in \operatorname{Hom}(V_n, W_n)$
- $\psi_n \in \operatorname{Ext}(V_n, W_{n-1})$, thought of as a map $V_n \to W_{n-1}[1]$ which travels down one degree.

There is a (fully faithful) embedding $\operatorname{Rep} Q \hookrightarrow \mathcal{D}^b(\operatorname{Rep} Q)$, in degree 0.

$\mathcal{D}^b(\mathrm{Rep}\, Q)$ is orientation-independent

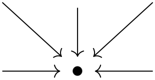
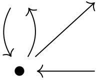
Theorem

If $\Gamma(Q) = \Gamma(Q')$ *is an acyclic graph, then* $\mathcal{D}^b(\mathrm{Rep}\, Q) \simeq \mathcal{D}^b(\mathrm{Rep}\, Q')$.

Since $\mathrm{Rep}\, Q \hookrightarrow \mathcal{D}^b(\mathrm{Rep}\, Q)$, we obtain embeddings $\mathrm{Rep}\, Q' \hookrightarrow \mathcal{D}^b(\mathrm{Rep}\, Q)$ for any Q' with the same underlying graph as Q, via these equivalences. We will be able to compute these equivalences explicitly.

Reflections on Rep Q

Let $x \in Q_0$ be a sink.

a sink:  not a sink: 

We get a new quiver $\sigma_x Q$ with $\Gamma(Q) = \Gamma(\sigma_x Q)$ by reversing all the arrows incident at x. There is a *reflection functor* $C_x^+ : \text{Rep}\, Q \to \text{Rep}\, \sigma_x Q$. On $V \in \text{Rep}\, Q$, it is defined by:

- $C_x^+ V(y) = V(y)$ for $x \neq y \in Q_0$
- $C_x^+ V(x)$ is the kernel of the sum of maps incident at x:

$$0 \to C_x^+ V(x) \to \bigoplus_{\substack{\alpha \in Q_1 \\ h\alpha = x}} V(t\alpha) \xrightarrow{\xi} V(x)$$

where ξ is the sum of the maps $V(\alpha) : V(t\alpha) \to V(x)$.

Reflections are not equivalences

Example (S_x is annihilated by C_x^+)

For any arrow with $h\alpha = x$, we have $t\alpha \neq x$ since we disallow loops. So $S_x(t\alpha) = 0$, and

$$\bigoplus_{\substack{\alpha \in Q_1 \\ h\alpha = x}} S_x(t\alpha) = 0.$$

Therefore the subspace $C_x^+(S_x) = 0$. For any other vertex $y \neq x$, $C_x^+ S_x(y) = S_x(y) = 0$. Thus $C_x^+ S_x = 0$. Similarly for C_x^-.

The reflection functors $C_x^+ : \operatorname{Rep} Q \to \operatorname{Rep} \sigma_x Q$ and $C_x^- : \operatorname{Rep} \sigma_x Q \to \operatorname{Rep} Q$ are not equivalences, because they both annihilate the 1-dimensional simple representation S_x.

An example reflection

Consider C_x^+ on D_4:

$$\begin{array}{ccccc} & & b & & \\ & & \downarrow & & \\ a & \longrightarrow & x & \longleftarrow & c \end{array}$$

Here are the reflections of a selection of indecomposables.

V	$C_x^+(V)$	V	$C_x^+(V)$
S_a	$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 1 & \xleftarrow{\sim} & 1 & \longrightarrow & 0 \end{array}$	$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 1 & \xrightarrow{\sim} & 1 & \longleftarrow & 0 \end{array}$	S_a
S_b	$\begin{array}{ccccc} & & 1 & & \\ & & \uparrow{\scriptstyle\sim} & & \\ 0 & \longleftarrow & 1 & \longrightarrow & 0 \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow{\scriptstyle\sim} & & \\ 0 & \longrightarrow & 1 & \longleftarrow & 0 \end{array}$	S_b
S_x	0	$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 1 & \xrightarrow{\sim} & 1 & \xleftarrow{\sim} & 1 \end{array}$	$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 1 & \xleftarrow{\sim} & 1 & \xrightarrow{\sim} & 1 \end{array}$
$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow{\scriptstyle\sim} & & \\ 1 & \xrightarrow{\sim} & 1 & \xleftarrow{\sim} & 1 \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & & \uparrow{\scriptstyle \mathrm{pr}_2} & & \\ 1 & \xleftarrow{\mathrm{pr}_1} & 2 & \xrightarrow{\mathrm{pr}_1+\mathrm{pr}_2} & 1 \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow{\scriptstyle \iota_2} & & \\ 1 & \xrightarrow{\iota_1} & 2 & \xleftarrow{\iota_1+\iota_2} & 1 \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & & \uparrow{\scriptstyle\sim} & & \\ 1 & \xleftarrow{\sim} & 1 & \xrightarrow{\sim} & 1 \end{array}$

Derived reflections are equivalences

The reflections C_x^+ and C_x^- induce derived reflections

$$\begin{aligned} RC_x^+ &: \mathcal{D}^b(\operatorname{Rep} Q) &\longrightarrow \mathcal{D}^b(\operatorname{Rep} \sigma_x Q) \\ LC_x^- &: \mathcal{D}^b(\operatorname{Rep} \sigma_x Q) &\longrightarrow \mathcal{D}^b(\operatorname{Rep} Q). \end{aligned}$$

Theorem

When Q is acyclic, RC_x^+ and LC_x^- are inverse equivalences.

To define RC_x^+ of an object $V = (V_n)_{n\in\mathbb{Z}}$, we find a bounded chain complex $\mathcal{I}$ whose terms are injective objects, and such that $\mathcal{I} \simeq V$ in $\mathcal{D}^b(\operatorname{Rep} Q)$. Then apply C_x^+ to $\mathcal{I}$, and compute cohomology to determine the components of $RC_x^+ V$.

Computation of a derived reflection

Example ($RC_x^+(S_x)$ on D_4)

We have an injective resolution of S_x:

$$0 \to S_x \to \left(E = \begin{array}{ccccc} & & 1 & & \\ & & \Big\downarrow\wr & & \\ 1 & \xrightarrow{\sim} & 1 & \xleftarrow{\sim} & 1 \end{array} \right) \to S_a \oplus S_b \oplus S_c \to 0$$

So $\mathcal{I} := (E \to S_a \oplus S_b \oplus S_c) \simeq S_x \in \mathcal{D}^b(\operatorname{Rep} D_4)$. Computing $C_x^+\mathcal{I}$ gives

$$\begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 1 & \longleftarrow & 2 & \longrightarrow & 1 \end{array} \hookrightarrow \begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 1 & \xleftarrow{\sim} & 1 & \longrightarrow & 0 \end{array} \oplus \begin{array}{ccccc} & & 1 & & \\ & & \uparrow\wr & & \\ 0 & \longleftarrow & 1 & \longrightarrow & 0 \end{array} \oplus \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & 1 & \xleftarrow{\sim} & 1 \end{array}$$

and the cohomology of this complex is $H^1(C_x^+\mathcal{I}) = S_x$, $H^i(C_x^+\mathcal{I}) = 0$ for $i \neq 1$. So

$$RC_x^+(S_x) = C_x^+\mathcal{I} \simeq H^1(C_x^+\mathcal{I})[-1] = S_x[-1]$$

since all the other cohomology pieces are 0. Here $[-1]$ denotes a shift in degree.

In fact, we always have $RC_x^+(S_x) = S_x[-1]$.

Reflections are transitive on acyclic underlying graphs

Theorem

If G is an acyclic graph, then there is a sequence of reflections between any two quivers Q and Q' with $\Gamma(Q) = G = \Gamma(Q')$.

Example (Reflections are transitive on orientations of D_4)