Mathematics is the art of giving the same name to different things.

Henri Poincaré

SYLOW THEOREMS FOR ∞-GROUPS

ISABEL LONGBOTTOM, 28 NOVEMBER 2023

Abstract. Taking the fundamental group gives an equivalence of categories between pointed topological spaces whose homotopy groups π_n for $n \neq 1$ all vanish, and groups. From this perspective, group theory can be viewed as the truncation of a more general theory of pointed connected spaces. Then the natural question is: to what extent can we do group theory in this new homotopical setting (where we don't require the higher homotopy groups to vanish)? In this talk, we will translate the Sylow theorems for finite groups to the context of finite ∞-groups, and use this to get a group-theoretic classification of finite nilpotent spaces, by analogy to the classification of finite nilpotent groups. We will also discuss the failure of normality for ∞-groups, as something of a cautionary tale.

Outline

(1) statement of classical Sylow theorems
(2) ∞-groups (and ∞-groupoids)
(3) translation of Sylow theorems to ∞-groups
(4) idea of proof, and one thing that goes wrong
(5) an application

These notes were prepared for a talk I gave at the Trivial Notions seminar at Harvard, a graduate student-led seminar designed to be accessible to all graduate students regardless of their particular mathematical interests. Essentially all of the content is drawn from the paper of the same name by Prasma and Schlank, [1].

1. Classical Sylow Theorems

Recall the following definition:

Definition 1.1. Let $H \subset G$ be finite groups, and p a prime number. H is called a *p-Sylow subgroup* of G if H is a p-subgroup of maximal possible order (i.e. $|H| = p^k$ where $|G| = m_p p^k$ with m_p coprime to p).

Note that p-Sylow subgroups can be trivial, if $p \nmid G$. Classically, the Sylow theorems are stated as follows.

Theorem 1.2 (Sylow Theorem)**.** *Let G be a finite group. For each prime p, let n_p denote the number of p-Sylow subgroups of G. Then*

(1) $n_p \equiv 1 \mod p$ and in particular, a p-Sylow subgroup exists.
(2) For fixed p, all p-Sylow subgroups are conjugate to one another.
(3) For a p-group H, any map $H \to G$ factors through some p-Sylow subgroup of G.
(4) With m_p be the index of a p-Sylow subgroup in G, we have $n_p \mid m_p$.

Remark 1.3. Some notes:

- It is immediate from 2 that all p-Sylow subgroups have the same order, and therefore the same index m_p in G. So 4 only depends on G, not a particular choice of p-Sylow subgroup in G.
- Our ∞-group version of this theorem will contain analogous statements to 1, 2, and 3. We *won't* get an analogous statement for 4, because we won't be able to come up with a sensible definition of index for ∞-groups.

2. What is an ∞-group?

The idea of this section is that we would like to be able to do group theory with topological spaces. A functorial way to associate a group with a (connected) topological space is via the fundamental group. We will generalise this by also incorporating higher homotopy groups.

Let G be a group. One can construct a space BG, its *classifying space*, with homotopy groups

$$\pi_n(BG) = \begin{cases} G & n = 1 \\ * & \text{else,} \end{cases}$$

at least when G is discrete. The amazing fact is that taking the fundamental group is left adjoint to the classifying space functor, when we restrict to pointed connected spaces. Or more simply, homotopy classes of pointed continuous maps $BG \to BH$ are in bijection with group homomorphisms $G \to H$.

Theorem 2.1. *There is an equivalence of categories between groups and pointed connected spaces whose higher homotopy groups $\pi_n, n \geq 2$ all vanish.*

Definition 2.2. We call a space *n-truncated* if π_k is trivial for all $k > n$.

So, groups are really the same thing as 1-truncated pointed connected spaces. We need *connected* here so that there is only one possible associated fundamental group, independent of the point at which we compute it. We need *pointed* so that maps between such objects agree.

Now, if we want to understand topological spaces, we can't restrict ourselves to those which are 1-truncated. Discarding all the higher homotopical information identifies spaces which we want to think of as distinct.

Example 2.3. The spaces S^2 and D^2 are both connected, so we can think of them as pointed connected by arbitrarily choosing a basepoint. Then both spaces have the same (trivial) fundamental group, but D^2 is contractible while S^2 is not, so they are not homotopically the same. We need information about the (not all trivial) higher homotopy groups of S^2 to distinguish them.

From now on we will use 1-group to mean group, where the '1' here refers to the fact that the corresponding space is 1-truncated. Then there is a natural generalisation where we allow some or all of the higher homotopy groups to be non-vanishing.

Definition 2.4. An n-group is a pointed connected n-truncated space. An ∞-group is a pointed connected space (we no longer require any of the higher homotopy groups to vanish). A pair of ∞-groups are isomorphic if there is a map between them inducing an isomorphism on every homotopy group (i.e. if they are weak homotopy equivalent) and similarly for n-groups.

There are many settings (e.g. disconnected spaces) where we need to talk about the fundamental *groupoid* rather than the fundamental group.

Definition 2.5. A (1-)groupoid is a category where all the morphisms are isomorphisms.

Definition/Theorem 2.6. A group is a pointed, connected groupoid (with pointed maps between groups).

The same relationship is true for n-groups and ∞-groups. We have the following technical definition.

Definition 2.7. The category of ∞-groupoids is obtained from the category of topological spaces by inverting weak homotopy equivalences. That is, we impose the condition that any map which induces an isomorphism on all homotopy groups is an equivalence of ∞-groupoids. (This is a localisation.) An ∞-group is then a pointed, connected ∞-groupoid, and a map of ∞-groups is a map of ∞-groupoids which respects the chosen basepoints.

Remark 2.8. We want ∞-groupoids to behave like the fundamental groupoid of a topological space, but preserve higher homotopical information. Definition 2.7 is in some sense the simplest thing we can do which achieves this.

You should think of an ∞-groupoid as a homotopy type, in the sense that it is a (nice) topological space up to homotopy. The term *space* is also used to mean ∞-groupoid.

Remark 2.9. In between 1-groupoids and ∞-groupoids, there is a notion of n-groupoid for each n. The n here means that we throw away all the information about homotopy groups above n. There are also truncation functors

$$\infty\text{-Gpd} \to n\text{-Gpd}$$

for each n, and compatible truncations $(n+1)$-Gpd $\to n$-Gpd. These are called Postnikov truncations. The same holds with 'groupoid' replaced everywhere by 'group', since the truncation of a pointed connected space is connected and naturally inherits the basepoint.

Remark 2.10. It is reasonable to call these things we have constructed ∞-groupoids because one can show that all the maps are invertible (in the same way that paths are invertible when forming the fundamental 1-groupoid).

3. Translation of Sylow Theorems to ∞-Groups

First we will need to make several definitions. We need to understand – in the context of ∞-groups – what Sylow subgroups should be, what finite means, and how to count the number of p-Sylow subgroups. The only thing we know how to do with ∞-groups is take their homotopy groups, so everything we need will be defined in terms of this.

Definition 3.1. An ∞-group is called *finite* if all its homotopy groups are finite. A finite ∞-group is called a *p-∞-group* if all its homotopy groups are p-groups. A map of finite ∞-groups $f : \mathbb{P} \to \mathbb{G}$ is called a *p-Sylow map* if for each $n \geq 1$, the induced map of homotopy groups

$$\pi_n(f) : \pi_n(\mathbb{P}) \to \pi_n(\mathbb{G})$$

is the inclusion of a p-Sylow subgroup.

Finally, we want to count the p-Sylow subgroups of an ∞-group $\mathbb{G}$. Let N_p denote the collection of p-Sylow maps $\mathbb{P} \to \mathbb{G}$. What kind of object is N_p? Well, the collection of all maps of p-∞-groups $\mathbb{H} \to \mathbb{G}$ is an ∞-groupoid, and we want the full subcategory spanned by maps which happen to be p-Sylow. Full subcategories of ∞-groupoids already have the property that all their morphisms are invertible, so these turn out to also be ∞-groupoids. So N_p comes equipped with the structure of an ∞-groupoid, i.e. it is a homotopy type.

But it turns out that N_p is actually discrete – it has the homotopy type of a finite disjoint union of points. So, we can really think of N_p as a number by counting the points. This is the number of p-Sylow subgroups.

Theorem 3.2 (∞-Sylow Theorem)**.** *Let $\mathbb{G}$ be a finite ∞-group. Fix a prime p, and let N_p be the ∞-groupoid of p-Sylow maps $\mathbb{P} \to \mathbb{G}$. Then*

(1) N_p is discrete and equivalent to the **set** *of p-Sylow subgroups of $\pi_1(\mathbb{G})$. In particular, $|\pi_0 N_p| \equiv 1 \mod p$ and there exists a p-Sylow map.*
(2) Any two p-Sylow maps are conjugate.
(3) For a finite p-∞-group $\mathbb{H}$, any map $\mathbb{H} \to \mathbb{G}$ factors through some p-Sylow map $\mathbb{P} \to \mathbb{G}$.

Remark 3.3. We said that N_p was finite discrete, i.e. as a homotopy type it is a finite union of points. We can think of it as the set consisting of these points. Then $|\pi_0 N_p|$ just counts the size of this set.

The fact that we have this theorem justifies our definition of a p-Sylow map of ∞-groups. We see that p-Sylow maps really are maximal (by 3 and 2), even though, unlike in the finite case, we didn't *define* them to be maximal.

If you have been paying attention, you'll notice that we haven't defined everything we need to state this theorem. Can someone tell me what is left to do?

Definition 3.4. Let $B\operatorname{aut}_* \mathbb{G}$ denote the automorphism ∞-groupoid of $\mathbb{G}$ at the base object (i.e. pointed automorphisms). Just as a 1-group acts on itself by conjugation, there is a canonical map $\operatorname{conj} : \mathbb{G} \to B\operatorname{aut}_* \mathbb{G}$ which we think of as conjugation.

Going back to our key example $\mathbb{G} = BG$ for a finite group G, the above conjugation induces

$$G = \pi_1(BG) = \pi_1(\mathbb{G}) \xrightarrow{\pi_1(\operatorname{conj})} \pi_1(B\operatorname{aut}_* \mathbb{G}) = \pi_1(B\operatorname{aut}_* BG) = \pi_1(B\operatorname{aut} G) = \operatorname{aut} G$$

and this agrees with our usual notion of conjugation. So this is really a generalisation of the conjugation map for 1-groups.

Definition 3.5. A pair of ∞-group maps $f : \mathbb{H} \to \mathbb{G}, f' : \mathbb{H}' \to \mathbb{G}'$ are *conjugate* if $\mathbb{H}$ and $\mathbb{H}'$ are isomorphic via a conjugation of $\mathbb{G}$. That is, there exist equivalences $\psi : \mathbb{G} \xrightarrow{\simeq} \mathbb{G}, \varphi : \mathbb{H} \xrightarrow{\simeq} \mathbb{H}$ making the diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\varphi,\simeq} & \mathbb{H}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathbb{G} & \xrightarrow[\psi,\simeq]{} & \mathbb{G} \end{array}$$

commute, and where $[\psi] \in \pi_0(\operatorname{aut}_* \mathbb{G}) = \pi_1(B\operatorname{aut}_* \mathbb{G})$ lies in the image of

$$\pi_1(\operatorname{conj}) : \pi_1(\mathbb{G}) \to \pi_1(B\operatorname{aut}_* \mathbb{G}).$$

4. Idea of proof

We will not give the proof of Theorem 3.2. However, in an attempt to indicate some of the flavour of the proof, we state the main technical result used and indicate how the theorem then follows.

Fix a finite ∞-group $\mathbb{G}$, a p-∞-group $\mathbb{H}$, and a map of ∞-groups $f : \mathbb{H} \to \mathbb{G}$.

Definition 4.1. We denote by N_f the ∞-groupoid spanned by factorisations of f as $\mathbb{H} \to \mathbb{P} \to \mathbb{G}$ with the second map being p-Sylow.

Why should this be an ∞-groupoid? It's the full sub-∞-groupoid of the ∞-category of all factorisations of f. In particular, when $f : * \to \mathbb{G}$ is the unique (pointed) map, then N_f is the space of all p-Sylow maps to $\mathbb{G}$.

Notation: Let $N_p = N_{*\to\mathbb{G}}$ be this space of p-Sylow maps to $\mathbb{G}$.

Aside 4.2. The ∞-category of all factorisations, $\operatorname{Fact}(f)$, can be computed as a pullback of ∞-categories via the diagram

$$\begin{array}{ccc} \operatorname{Fact}(f) & \longrightarrow & \{f\} \\ \downarrow & \lrcorner & \downarrow \\ \infty\text{-Grp}^{\Delta^2} & \xrightarrow[{[02]_*}]{} & \infty\text{-Grp}^{\Delta^1} \end{array}.$$

Now, applying the (Postnikov) truncation functors which we discussed earlier, we obtain a tower

$$N_f \to \ldots \to N_f[n+1] \to N_f[n] \to \ldots \to N_f[1]$$

where $N_f[n]$ denotes the truncation so that

$$\pi_k(N_f[n]) = \begin{cases} \pi_k(N_f) & k \leq n \\ 0 & k > n. \end{cases}$$

Note also that $N_f = \lim N_f[n]$.

Lemma 4.3. *For $n \geq 1$, the map $N_f[n+1] \to N_f[n]$ is an equivalence (its fibres are all contractible). Hence the ∞-groupoid N_f is equivalent to the 1-groupoid $N_f[1]$, and in particular discrete.*

Aside 4.4. We can reinterpret this lemma as follows. Given a diagram

$$\begin{array}{ccccc} \mathbb{H} & & \xrightarrow{\quad f \quad} & & \mathbb{G} \\ \downarrow & & & & \downarrow \\ \mathbb{H}[n] & \longrightarrow & \mathbb{P}_n & \longrightarrow & \mathbb{G}[n] \end{array}$$

where the bottom row is an element of $N_f[n]$, we can consider the ∞-groupoid S of completions of the diagram to

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & \mathbb{P} & \longrightarrow & \mathbb{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}[n] & \longrightarrow & \mathbb{P}_n & \longrightarrow & \mathbb{G}[n] \end{array}$$

where $\mathbb{P} \to \mathbb{G}$ is a p-Sylow map and $\mathbb{P} \to \mathbb{P}_n$ is the n-Postnikov truncation. Then S is contractible, i.e. there is essentially a unique way to choose the map $\mathbb{P} \to \mathbb{G}$ to be compatible with the diagram.

This reinterpretation is the version of the lemma that is proved in [1].

Now we are ready to prove Theorem 3.2 using Lemma 4.3.

Proof. From Lemma 4.3, we know $N_f \simeq N_f[1]$. The RHS has no nontrivial homotopy groups above π_1, which means N_f is discrete, and equivalent to the *set* of factorisations

$$\pi_1(f) : \pi_1(\mathbb{H}) \to P \to \pi_1(\mathbb{G})$$

where $P \to \pi_1(\mathbb{G})$ is a p-Sylow subgroup inclusion (of ordinary groups). But this is the same as the set of p-Sylow subgroups of $\pi_1(\mathbb{G})$ that contain the image of $\pi_1(\mathbb{H})$. This set is nonempty because the image of $\pi_1(\mathbb{H})$ is a p-subgroup of $\pi_1(\mathbb{G})$, so must be contained in a maximal p-subgroup. This proves 3.

Now for 1 we consider $f : * \to \mathbb{G}$ and the corresponding ∞-groupoid N_p. We see that N_p is discrete and equivalent to the set of p-Sylow subgroups of $\pi_1(\mathbb{G})$.

Note 2 also follows essentially immediately from Lemma 4.3, also by considering N_p. □

Example 4.5. When $\mathbb{G} = BG$ for a finite group G, we can recover the classical Sylow theorem 1.2 from Theorem 3.2. This is essentially immediate by applying π_1 everywhere.

Despite everything we have seen, the analogy between Theorem 3.2 and the classical Sylow theorems is not perfect. We mentioned previously that the classical Sylow theorem 1.2 includes an additional condition 4 that $n_p \mid m_p$, the index of any Sylow subgroup. This is not replicated for ∞-groups because we don't have a sensible notion of index in this setting.

But there is actually a much more serious failing, which stems from the following. A normal subgroup of a 1-group can be defined in two equivalent ways: either as the kind of subgroup by which we can quotient, or as subgroup whose conjugates all recover the subgroup itself. In the setting of ∞-groups, both of these definitions make sense, but they are no longer equivalent.

Definition 4.6. A map of ∞-groups $f : \mathbb{P} \to \mathbb{G}$ is called *normal* if there exists a fibre sequence $\mathbb{P} \to \mathbb{G} \to \mathbb{Q}$. We think of $\mathbb{Q}$ as some sort of quotient of $\mathbb{G}$ by f, denoted $\mathbb{Q} = \mathbb{G}//\mathbb{P}$.

Note that a fibre sequence induces a long exact sequence on homotopy groups.

Here is the problem: it can occur that $\mathbb{G}$ has a unique p-Sylow subgroup $\mathbb{P} \to \mathbb{G}$, but this subgroup is not normal. In such case it is still true that all of the conjugates of $\mathbb{P}$ are $\mathbb{P}$ itself, but this is not enough to guarantee that an appropriate quotient of $\mathbb{G}$ by $\mathbb{P}$ exists.

Example 4.7. Let $A = \{0, \pm 1\}$ be the $(\mathbb{Z}/2)$-module with 3 elements and non-trivial action. Let $X = K(A, 2)//\mathbb{Z}/2$. We interpret this as follows: $\pi_0(\mathbb{Z}/2) = \mathbb{Z}/2$ i.e. 2 points, and $\pi_n(\mathbb{Z}/2) = 0$ for $n \geq 1$. Since $K(A, 2)$ is an Eilenberg-MacLane space, $\pi_2(K(A, 2)) = A = \mathbb{Z}/3$ is a group with 3 elements, and all its other homotopy groups are trivial. Then it follows from the long exact sequence that $\pi_2(X) = A$ and $\pi_1(X) = \mathbb{Z}/2$, with other homotopy groups trivial.

Since $\pi_1(X) = \mathbb{Z}/2$ is abelian, it has a unique 2-Sylow subgroup $\mathbb{Z}/2$. By Theorem 3.2, there is therefore a unique 2-Sylow map $B\mathbb{Z}/2 \to X$, which is an isomorphism on π_1. We claim that this map is not normal.

Suppose it were normal, with quotient Y. Then from the long exact sequence corresponding to $B\mathbb{Z}/2 \to X \to Y$, we find that $\pi_2(Y) = \mathbb{Z}/3$ and all other homotopy groups are trivial. This means $Y = K(\mathbb{Z}/3, 2)$ is the only possible choice for the quotient. But there is no fibre sequence diagram

$$B\mathbb{Z}/2 \to K(A, 2)//\mathbb{Z}/2 \to K(\mathbb{Z}/3, 2),$$

by the functoriality of the π_1-action.

It is a general fact that $\pi_1(Z)$ acts on all the higher homotopy groups. For any map $f : W \to Z$, the induced map $\pi_n(f) : \pi_n(W) \to \pi_n(Z)$ is compatible with the π_1-actions. We know that the map $\pi_2(X) \to \pi_2(Y)$ is an isomorphism $A \to A$, because $\pi_2(B\mathbb{Z}/2) = 0$. The π_1-action on the left is the nontrivial action of $\pi_1(X) = \mathbb{Z}/2$ on A, while the action on the right is trivial since $\pi_1(K(\mathbb{Z}/3, 2)) = 0$. Hence compatibility with the π_1-action is impossible.

The relevant section of the long exact sequence on homotopy groups is

$$0 = \pi_2(B\mathbb{Z}/2) \to \pi_2(X) \to \pi_2(Y) \to \pi_1(B\mathbb{Z}/2) \xrightarrow{\simeq} \pi_1(X) \to \pi_1(Y) \to \pi_0(B\mathbb{Z}/2) = 0.$$

The converse is still true: a normal map $\mathbb{P} \to \mathbb{G}$ has no nontrivial conjugates. So normality (i.e. having a quotient) for ∞-groups is strictly stronger than the property that all conjugates recover the original map.

5. One application

In this section, we present another classical result and its ∞-group analog. The proof is by a direct application of the Sylow Theorem for ∞-groups, and induction along the Postnikov tower.

Definition 5.1. An ∞-group $\mathbb{G}$ is *nilpotent* if $\pi_1\mathbb{G}$ is nilpotent and acts nilpotently on $\pi_n\mathbb{G}$ for every $n \geq 2$.

The concept of nilpotence for spaces is actually very important and arises in many places in homotopy theory.

Here is the classical theorem characterising nilpotent finite groups.

Theorem 5.2. *Let G be a finite group. The following are equivalent:*

(1) G is nilpotent.
(2) G is isomorphic to the product of its Sylow subgroups.
(3) All Sylow subgroups of G are normal.

As a corollary, finite nilpotent groups are precisely (finite) products of p-groups. Now, we have an ∞-group analog:

Theorem 5.3. *Let $\mathbb{G}$ be a finite ∞-group, and $G = \pi_1(\mathbb{G})$. The following are equivalent:*

(1) $\mathbb{G}$ is nilpotent.
(2) $\mathbb{G} \simeq \prod_{p|G} \mathbb{P}_p$, each $\mathbb{P}_p \to \mathbb{G}$ being a p-Sylow map.
(3) All p-Sylow maps $\mathbb{P}_p \to \mathbb{G}$ are normal.

Here we mean normal in the strong sense, that is an appropriate quotient $\mathbb{G}//\mathbb{P}_p$ exists. Just requiring that there be a unique p-Sylow map for each prime p is strictly weaker than either of the three conditions in the theorem, although it is equivalent in the 1-group case. Also note that requiring all p-Sylow maps to $\mathbb{G}$ to be normal is the same thing (via the Sylow theorem) as requiring that for each prime $p \mid G$, there exists some normal p-Sylow map to $\mathbb{G}$.

Proof. First, $2 \implies 3$ is clear via the projection map out of the product.

For $3 \implies 1$, notice that each p-Sylow subgroup of $G = \pi_1(\mathbb{G})$ is normal and thus by the classical version of the theorem, $\pi_1(\mathbb{G})$ is nilpotent. Also, by an induction on the LES of $\mathbb{P}_p \to \mathbb{G} \to \mathbb{G}//\mathbb{P}_p$, we know that $\pi_n(G) \cong \prod_{p|G} \pi_n(\mathbb{P}_p)$ for every $n \geq 1$. From the LES, we can also deduce that $\pi_1(\mathbb{P}_p)$ acts trivially on the prime-to-p part of $\pi_n(\mathbb{G})$, i.e. the only part of the action of $\pi_1(\mathbb{G})$ on $\pi_n(\mathbb{G})$ which can be nontrivial comes from each $\pi_1(\mathbb{P}_p)$ acting on the corresponding $\pi_n(\mathbb{P}_p)$. But this action is nilpotent because $\pi_1(\mathbb{P}_p)$ is a p-group and $\pi_n(\mathbb{P}_p)$ has p-power order.

Finally, we must show $1 \implies 2$. Argue by induction on the (Postnikov) n-truncations of $\mathbb{G}$. If $\mathbb{G}$ is already 1-truncated, the result is immediate from Theorem 5.2. Now assume the product formula $\mathbb{G}[n] \simeq \prod_{p|G} \mathbb{P}_p[n]$ holds for any nilpotent n-truncated group, and consider some $\mathbb{G}[n+1]$. We have a pullback square

$$\begin{array}{ccc} \mathbb{G}[n+1] & \longrightarrow & B\pi_1(\mathbb{G}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{G}[n] & \longrightarrow & K(\pi_{n+1}(\mathbb{G}), n+2)//\pi_1(\mathbb{G}) \end{array}$$

which you can think of as extending $\mathbb{G}[n]$ up to $\mathbb{G}[n+1]$ by adding on the homotopy group $\pi_{n+1}(\mathbb{G})$. The three terms other than $\mathbb{G}[n+1]$ all split as products in the desired way ($\mathbb{G}[n]$ by the induction hypothesis) and the maps between them respect the product structure. So the pullback $\mathbb{G}[n+1]$ splits as a product. Explicitly, the whole diagram decomposes as a product of diagrams of the form

$$\begin{array}{ccc} \mathbb{P}_p[n+1] & \longrightarrow & B\pi_1(\mathbb{P}_p) \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{P}_p[n] & \longrightarrow & K(\pi_{n+1}(\mathbb{P}_p), n+2)//\pi_1(\mathbb{P}_p) \end{array}$$

and so we find that $\mathbb{G} \simeq \prod_{p|G} \mathbb{P}_p[n+1]$.

An arbitrary finite ∞-group is the limit of its n-truncations, so the statement follows here too. $\square$

References

[1] M. Prasma and T. Schlank, "Sylow theorems for ∞-groups," *Topology and its Applications*, vol. 222, pp. 121–138, 2017.