Worksheet

Q1. Let X be a space covered by two open sets A and B, and write $C_\bullet(A+B)$ for the subcomplex $C_\bullet(A)+C_\bullet(B)$ of $C_\bullet(X)$. Let

$$i : A\cap B \hookrightarrow A, \qquad j : A\cap B \hookrightarrow B, \qquad k : A \hookrightarrow X, \qquad l : B \hookrightarrow X$$

be the four inclusions, and define φ and ψ by

$$\varphi(x) = (i_*x,\ -j_*x), \qquad \psi(y,z) = k_*y + l_*z.$$

$$0 \longrightarrow C_\bullet(A\cap B) \xrightarrow{\ \varphi\ } C_\bullet(A)\oplus C_\bullet(B) \xrightarrow{\ \psi\ } C_\bullet(A+B) \longrightarrow 0$$

Show that this sequence of chain complexes is short exact. Once you have done that, what does it give us?

Q2. (a) Write down an open cover of the sphere S^k and the corresponding Mayer–Vietoris sequence.

(b) Compute $H_n(S^k)$.

Excision

For $X = A \cup B$ with A, B open, we want to show that the map $H_n(B,\, A \cap B) \to H_n(X, A)$ induced by the inclusion of pairs is an isomorphism.

Q3. (a) Let's formalise our intuition and simplify the chain complex computing $H_n(B,\, A \cap B)$. Show that

$$\frac{C_\bullet(B)}{C_\bullet(A \cap B)} \;\cong\; \frac{C_\bullet(A) + C_\bullet(B)}{C_\bullet(A)}.$$

(b) Part (a) gives us a short exact sequence of chain complexes relating $C_\bullet(A) + C_\bullet(B)$, $C_\bullet(B)/C_\bullet(A \cap B)$ and $C_\bullet(A)$. Write it down.

Q4. (a) Compare your short exact sequence from Q3 to the one for the pair (X, A).

(b) Apply the snake lemma to your diagram.

(c) Explain why $H_n(A + B) \to H_n(X)$ is an isomorphism.

Q5. Show that any time you have four isomorphisms in a map of long exact sequences of abelian groups like this, the fifth (central) map is also an isomorphism.

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\cong \big\downarrow \alpha & & \cong \big\downarrow \beta & & \big\downarrow \gamma & & \cong \big\downarrow \delta & & \cong \big\downarrow \varepsilon \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

Both rows are exact and all four squares commute.

(a) Prove that γ is surjective if β and δ are surjective and ε is injective.

(b) Prove that γ is injective if β and δ are injective and α is surjective.

Q6. Conclude that $H_n(B,\, A \cap B) \cong H_n(X, A)$.